1. (a) By Hölder's inequality, if \( \phi \in C_c^\infty(U) \) then integration against \( \phi \) is an element of \( (L^p)^* \). Since convergence in \( L^p \) implies weak convergence, \( \lim_{n \to \infty} \int f_n \phi = \int f \phi \). This shows that \( (f_n)_{n=1}^\infty \) converges to \( f \) in \( \mathcal{D}'(U) \).

(b) If \( \phi \in C_c^\infty(U) \) then the support of \( \phi \) is bounded, and \( \phi \) itself is bounded, so \( |g\phi| \in L^1 \). Since \( |f_n\phi| \leq g|\phi| \) for all \( n \in \mathbb{N} \), the dominated convergence theorem implies that \( \lim_{n \to \infty} \int f_n \phi = \int f \phi \).

(c) The sequence \( (n\chi_{(0, \frac{1}{n})})_{n=1}^\infty \) converges to 0 pointwise, but to \( \delta \neq 0 \) in \( \mathcal{D}'(\mathbb{R}) \) (by Proposition 9.1).

10. By Theorem 2.49, if \( R \in (0, \infty) \) and \( f \neq 0 \) then

\[
\int_{B_R(0)} |f| \leq \int_0^R \int_{S^{n-1}} |f(rx)| r^{n-1} d\sigma(x) dr = \int_0^R \int_{S^{n-1}} |f(x)| r^{-1} dr d\sigma(x) = \int_{S^{n-1}} |f(x)| \cdot \infty d\sigma(x) = \infty,
\]

so \( f \) is not locally integrable near 0. If \( \phi \in C_c^\infty \) is supported on \( B_R(0) \) and \( \varepsilon \in (0, R) \) then

\[
\int_{B_{\epsilon}(0)} f \phi = \int_\varepsilon^R \int_{S^{n-1}} f(rx)\phi(rx)r^{n-1} d\sigma(x) dr = \int_\varepsilon^R \int_{S^{n-1}} f(x)(\phi(rx)-\phi(0))r^{-1} d\sigma(x) dr = \int_{B_R(0)\setminus B_{\epsilon}(0)} f(\phi-\phi(0))
\]

because \( \int_{S^{n-1}} f d\sigma = 0 \). By the mean value theorem there exists \( M \in (0, \infty) \) such that \( |\phi(x) - \phi(0)| \leq M|x| \) for all \( x \in \mathbb{R}^n \), in which case

\[
\int_{B_R(0)} |f||\phi - \phi(0)| \leq \int_0^R \int_{S^{n-1}} |f(rx)| |rx|r^{n-1} d\sigma(x) dr = \int_0^R \int_{S^{n-1}} |f(x)| M dr d\sigma(x) = MR \int_{S^{n-1}} |f| dr,
\]

so by the dominated convergence theorem \( \langle PV(f), \phi \rangle = \int_{B_R(0)} f(\phi - \phi(0)) \). In particular \( PV(f) \) is well-defined. It is clearly linear. If \( K \subseteq B_R(0) \) is compact and \( (\phi_n)_{n=1}^\infty \) a sequence in \( C_c^\infty(K) \) which converges to \( \phi \in C_c^\infty(K) \), then

\[
\lim_{n \to \infty} \langle PV(f), \phi_n \rangle = \lim_{n \to \infty} \int_{B_R(0)} f(\phi_n - \phi_n(0)) = \int_{B_R(0)} f(\phi - \phi(0)) = \langle PV(f), \phi \rangle,
\]

by the dominated convergence theorem. This shows that \( PV(f) \) is continuous, hence a distribution. It agrees with \( f \) on \( \mathbb{R}^n \setminus \{0\} \): if \( \phi \in C_c^\infty \) is supported on \( B_R(0) \setminus \{0\} \) then

\[
\langle PV(f), \phi \rangle = \int_{B_R(0)} f(\phi - \phi(0)) = \int_{B_R(0)} f \phi = \int f \phi = \langle f, \phi \rangle.
\]

If \( \phi \in C_c^\infty \) is supported on \( B_R(0) \) and \( r \in (0, \infty) \) then \( \phi \circ S_r^{-1} \in C_c^\infty(B_{rR}(0)) \) and hence (by Theorem 2.44)

\[
\langle PV(f) \circ S_r, \phi \rangle = |\det(S_r)|^{-1} \langle PV(f), \phi \circ S_r^{-1} \rangle = |\det(S_r^{-1})| \int_{B_{rR}(0)} f(\phi \circ S_r^{-1} - \phi(S_r^{-1}(0))) = \int_{B_R(0)} (f \circ \phi)(\phi - \phi(0)).
\]

Clearly \( f \circ S_r = r^n f \), so \( PV(f) \circ S_r = r^{-n} PV(f) \), which means \( PV(f) \) is homogeneous of degree \( -n \).