

The Goat and Silo Problem

Problem. A goat is tethered to a cylindrical silo of radius r with a rope of length l , $l \leq \pi r$. (See Figure 1.) Find the area of the region the goat can graze.

Solution. The area the goat can graze is $\frac{\pi l^2}{2}$ plus twice the "evolute area" (see Figure 2). To find the evolute area, we evaluate the line integral $\frac{1}{2} \int_C -y dx + x dy$ along its boundary curve C . To do this, we observe that $C = C_1 \cup C_2 \cup C_3$, where C_1 is the circular arc along the silo, C_2 is the line tangent to the silo at the point where the goat is tethered, and C_3 is the evolute (Figure 2). The curves C_1 , C_2 , and C_3 can be parametrized as follows. (Note that the parametrizations describe a counterclockwise path.)

$$\begin{aligned} C_1 : \mathbf{r}(t) &= r \left(\cos \left(\frac{l}{r} - t \right), \sin \left(\frac{l}{r} - t \right) \right), \\ C_2 : \mathbf{r}(t) &= r(1, t), \\ C_3 : \mathbf{r}(t) &= r \left(\cos t - \left(\frac{l}{r} - t \right) \sin t, \sin t + \left(\frac{l}{r} - t \right) \cos t \right), \end{aligned}$$

where $t \in [0, \frac{l}{r}]$. By additivity, the line integral around C is the sum of the line integrals along C_1 , C_2 , and C_3 .

$$\begin{aligned} \frac{1}{2} \int_{C_1} -y dx + x dy &= \frac{r^2}{2} \int_0^{\frac{l}{r}} \left[-\sin^2 \left(\frac{l}{r} - t \right) - \cos^2 \left(\frac{l}{r} - t \right) \right] dt \\ &= -\frac{r^2}{2} \int_0^{\frac{l}{r}} dt \\ &= -\frac{lr}{2} \\ \frac{1}{2} \int_{C_2} -y dx + x dy &= \frac{r^2}{2} \int_0^{\frac{l}{r}} dt \\ &= \frac{lr}{2} \\ \frac{1}{2} \int_{C_3} -y dx + x dy &= \frac{r^2}{2} \int_0^{\frac{l}{r}} \left\{ \left[\sin t + \left(\frac{l}{r} - t \right) \cos t \right] \left(\frac{l}{r} - t \right) \cos t - \right. \\ &\quad \left. \left[\cos t - \left(\frac{l}{r} - t \right) \sin t \right] \left(\frac{l}{r} - t \right) \sin t \right\} dt \\ &= \frac{r^2}{2} \int_0^{\frac{l}{r}} \left(\frac{l}{r} - t \right)^2 dt \\ &= \frac{l^3}{6r} \\ \frac{1}{2} \int_C -y dx + x dy &= -\frac{lr}{2} + \frac{lr}{2} + \frac{l^3}{6r} = \frac{l^3}{6r} \end{aligned}$$

Thus, the total area A the goat can graze is given by

$$A = \frac{\pi l^2}{2} + 2 \frac{l^3}{6r} = \frac{\pi l^2}{2} + \frac{l^3}{3r}.$$

The reason for the restriction $l \leq \pi r$ is illustrated in Figure 3. For another approach to this problem, see

M. E. Hoffman, The Bull and the Silo: An Application of Curvature, *American Mathematical Monthly*, January 1998, pages 55-58.

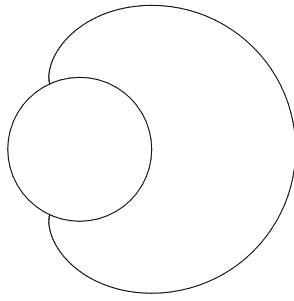


Figure 1: Goat's Grazing Area

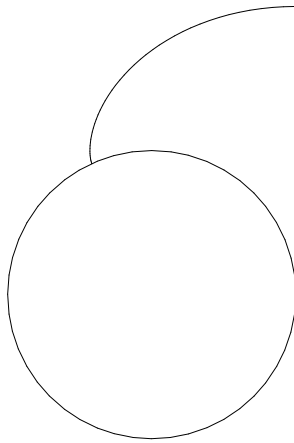


Figure 2: Evolute Area

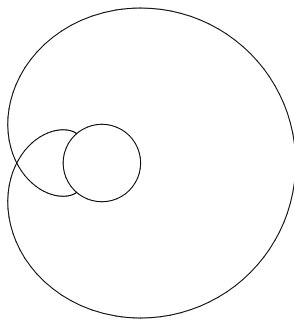


Figure 3: Goat's Grazing Area: $l > \pi r$