Math 142A Homework Assignment 4
Due Wednesday, November 8

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at x_0 with $f(x_0) > 0$. Prove that there is a natural number n for which $f(x) > 0$ for all x in the interval $I := (x_0 - 1/n, x_0 + 1/n)$.

2. A function $f : D \to \mathbb{R}$ is said to be a *Lipschitz function* if there is a $C \geq 0$ such that $|f(u) - f(v)| \leq C\,|u - v|$ for all u, v in D. Prove that a Lipschitz function is continuous.

3. Let $f : \mathbb{R} \to \mathbb{R}$ have the property that $f(u + v) = f(u) + f(v)$ for all u and v.
   (a) Let $m := f(1)$. Prove that $f(x) = mx$ for all rational numbers x.
   (b) Prove that if f is continuous, then $f(x) = mx$ for all x.

4. Let S be a nonempty set of real numbers that is *not* sequentially compact. Prove that there is an unbounded sequence in S or there is a sequence in S that converges to a point x_0 which is not in S.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) > 0$ and $f(1) = 0$. Prove that there is an x_0 in $(0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for all x in $[0, x_0)$; that is, there is a smallest point in the interval $[0, 1]$ at which f attains the value 0.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose image $f(\mathbb{R})$ is bounded. Prove that there is a solution to the equation $f(x) = x$.

7. Let $f : [a, b] \to \mathbb{R}$ be continuous. Given a natural number k, let $x_1, \ldots, x_k$ be points in $[a, b]$. Prove that there is a point z in $[a, b]$ at which

$$f(z) = \frac{f(x_1) + \cdots + f(x_k)}{k}.$$

   [Note: As $k \to \infty$, this becomes the mean value theorem for integrals (Theorem 6.26).]

8. Given $f : [0, 1] \to \mathbb{R}$ continuous such that $f([0, 1]) \subseteq \mathbb{Q}$. Show that f is a constant function.

9. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be uniformly continuous functions. Define the product function $fg : D \to \mathbb{R}$ by $(fg)(x) := f(x)g(x)$.
   (a) Show that fg need not be uniformly continuous.
   (b) Prove that if f and g are also bounded, then fg is uniformly continuous.
   Hint: Write $f(u)g(u) - f(v)g(v) = f(u)\,[g(u) - g(v)] + g(v)\,[f(u) - f(v)]$.

10. A function $f : D \to \mathbb{R}$ is called a *Lipschitz function* if there is a $C \geq 0$ such that $|f(u) - f(v)| \leq C\,|u - v|$ for all $u, v \in D$. Prove that if f is a Lipschitz function, then f is uniformly continuous.