Math 142A Homework Assignment 6
Due Wednesday, November 22

1. Show that there does not exist a strictly increasing function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(\mathbb{Q}) = \mathbb{R}$.

2. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and one-to-one and that $f(a) < f(b)$.

   (a) Let c be any point in the open interval (a, b). Prove that $f(a) < f(c) < f(b)$.

   (b) Prove that f is strictly increasing.

3. A point x_0 in D is said to be an isolated point of D if there is an $r > 0$ such that $D \cap (x_0 - r, x_0 + r) = \{x_0\}$; that is, the only point of D in $(x_0 - r, x_0 + r)$ is x_0 itself.

   (a) Prove that a point x_0 in D is either an isolated point or a limit point of D.

   (b) Suppose that x_0 is an isolated point of D. Prove that every function $f : D \to \mathbb{R}$ is continuous at x_0.

   (c) Prove that if x_0 is a limit point of D, then $f : D \to \mathbb{R}$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

4. Let $\{x_n\}$ be a bounded sequence such that $x_m \neq x_n$ whenever $m \neq n$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $f(x_n) = 0$ for every index n. Show that there is a point x_0 at which $f(x_0) = 0$ and $f'(x_0) = 0$.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and monotonically increasing. Show that $f'(x) \geq 0$ for all x.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Determine the value of the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h}.$$

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $f(x + y) = f(x)f(y)$ for all x and y.

   (a) Show that $f'(x) = f'(0)f(x)$.

   (b) What is $f(0)$? Justify your answer.