**Name:** ______________________________ **PID:** ________________

**TA:** _______________________ **Sec. No:** _____ **Sec. Time:** _____

**Math 142A**
**Final Examination**
**March 18, 2011**

*Turn off and put away your cell phone.*
*No calculators or any other electronic devices are allowed during this exam.*
*You may use one page of notes, but no books or other assistance on this exam.*
*Read each question carefully, answer each question completely, and show all of your work.*
*Write your solutions clearly and legibly; no credit will be given for illegible solutions.*
*If any question is not clear, ask for clarification.*

| # | Points | Score |
|---|---|---|
| **1** | 12 | |
| **2** | 6 | |
| **3** | 12 | |
| **4** | 6 | |
| **5** | 6 | |
| **6** | 8 | |
| **$\Sigma$** | 50 | |

1. (12 points) Define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = 2 \\ x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \text{ for } n \geq 1 \end{cases}$$

(a) Show that $\{x_n\}$ is bounded below by $\sqrt{2}$. (Hint: show that $x_n^2 - 2 > 0$ for every index $n$.)

(b) Show that $\{x_n\}$ decreases monotonically.

(c) Show that $x_n \to \sqrt{2}$.

2. (6 points) The sequential compactness theorem asserts that for real numbers $a < b$, the interval $[a, b]$ is sequentially compact. However, not every sequentially compact set is an interval.

   Show that a finite set of points $\mathcal{F} = \{a_1, a_2, \ldots, a_N\}$ is sequentially compact.

3. (12 points) Show that an arbitrary sequentially compact set $S$ (not necessarily an interval) is closed and bounded.

4. (6 points) Let $\{a_n\}$ be a convergent sequence with $a_n \geq 0$ for every index $n$, and let $a = \lim_{n\to\infty} a_n$. Is $a \geq 0$? If so, prove it; if not, exhibit a counterexample.

5. (6 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if and only if

$$f(-x) = -f(x) \quad \text{for all } x,$$

and $f : \mathbb{R} \to \mathbb{R}$ is called *even* if and only if

$$f(-x) = f(x) \quad \text{for all } x.$$

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and even, then $f' : \mathbb{R} \to \mathbb{R}$ is odd.

6. (8 points) Recall that a function $f : D \to \mathbb{R}$ is **C-Lipschitz** provided $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in D$.

   (a) Prove that if a differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a derivative bounded by $C$ (that is, $|f'(x)| \leq C$ for all $x$), then $f$ is $C$-Lipschitz.

   (b) Show that the absolute-value function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is 1-Lipschitz. (Note: This shows that a $C$-Lipschitz function need not be differentiable.)