Cylindrical Coordinates

Cylindrical coordinates are related to rectangular coordinates as follows.

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} & \qquad x &= \rho\cos\theta \\ \tan\theta &= \frac{y}{x} & \qquad y &= \rho\sin\theta \\ z &= z & \qquad z &= z \end{aligned}$$

The cylindrical coordinate vectors are defined as

$$\begin{aligned} \mathbf{e}_\rho &:= \frac{1}{|\nabla\rho|}\nabla\rho \\ \mathbf{e}_\theta &:= \frac{1}{|\nabla\theta|}\nabla\theta \\ \mathbf{e}_\mathbf{z} &:= \frac{1}{|\nabla z|}\nabla z \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{e}_\rho &= \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\,\mathbf{j} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \\ \mathbf{e}_\theta &= -\frac{y}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{j} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \\ \mathbf{e}_\mathbf{z} &= \mathbf{k} \end{aligned}$$

The inverse relationship is as follows.

$$\begin{aligned} \mathbf{i} &= \cos\theta\,\mathbf{e}_\rho - \sin\theta\,\mathbf{e}_\theta \\ \mathbf{j} &= \sin\theta\,\mathbf{e}_\rho + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{k} &= \mathbf{e}_\mathbf{z} \end{aligned}$$

It is worth noting that the above computations also imply the following.

$$\begin{aligned} \frac{\partial\rho}{\partial x} &= \cos\theta \\ \frac{\partial\rho}{\partial y} &= \sin\theta \\ \frac{\partial\theta}{\partial x} &= -\frac{1}{\rho}\sin\theta \\ \frac{\partial\theta}{\partial y} &= \frac{1}{\rho}\cos\theta \end{aligned}$$

The position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is written

$$\mathbf{R} = \rho\,\mathbf{e}_\rho + z\,\mathbf{e_z}. \quad \text{(cylindrical coordinates)}$$

If $\mathbf{R} = \mathbf{R}(t)$ is a parameterized curve, then $\frac{d\mathbf{R}}{dt} = \frac{d\rho}{dt}\mathbf{e}_\rho + \rho\frac{d\mathbf{e}_\rho}{dt} + \frac{dz}{dt}\mathbf{e_z}$. Since $\mathbf{e}_\rho = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, it follows that $\frac{d\mathbf{e}_\rho}{dt} = \frac{d\theta}{dt}\,\mathbf{e}_\theta$. Thus,

$$\frac{d\mathbf{R}}{dt} = \frac{d\rho}{dt}\,\mathbf{e}_\rho + \rho\frac{d\theta}{dt}\,\mathbf{e}_\theta + \frac{dz}{dt}\,\mathbf{e_z}$$

Hence, $d\mathbf{R} = d\rho\,\mathbf{e}_\rho + \rho\,d\theta\,\mathbf{e}_\theta + dz\,\mathbf{e_z}$ and it follows that the element of volume in cylindrical coordinates is given by

$$dV = \rho\,d\rho\,d\theta\,dz$$

If $f = f(x, y, z)$ is a scalar field (that is, a real-valued function of three variables), then

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

If we view x and y as functions of ρ and θ and apply the chain rule, we obtain

$$\nabla f = \left(\frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

Writing this in terms of ρ, θ, and the cylindrical coordinate vectors yields

$$\nabla f = \left(\cos\theta\frac{\partial f}{\partial \rho} - \frac{1}{\rho}\sin\theta\frac{\partial f}{\partial \theta}\right)(\cos\theta\,\mathbf{e}_\rho - \sin\theta\,\mathbf{e}_\theta) + \left(\sin\theta\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\cos\theta\frac{\partial f}{\partial \theta}\right)(\sin\theta\,\mathbf{e}_\rho + \cos\theta\,\mathbf{e}_\theta) + \frac{\partial f}{\partial z}\,\mathbf{e_z}.$$

Simplifying, we obtain the result

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e_z}.$$

If $\mathbf{F} = \mathbf{F}(x, y, z)$ is a vector field (that is, a vector-valued function of three variables), then we can write

$$\begin{aligned}\mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \\ &= (\cos\theta F_1 + \sin\theta F_2)\mathbf{e}_\rho + (-\sin\theta F_1 + \cos\theta F_2)\mathbf{e}_\theta + F_3\mathbf{e_z}\end{aligned}$$

Thus, $\mathbf{F} = F_\rho\,\mathbf{e}_\rho + F_\theta\,\mathbf{e}_\theta + F_z\,\mathbf{e_z}$, where

$$\begin{aligned}F_\rho &= \cos\theta\ F_1 + \sin\theta\ F_2 & \qquad F_1 &= \cos\theta\ F_\rho - \sin\theta\ F_\theta \\ F_\theta &= -\sin\theta\ F_1 + \cos\theta\ F_2 & \qquad F_2 &= \sin\theta\ F_\rho + \cos\theta\ F_\theta \\ F_z &= F_3 & \qquad F_3 &= F_z\end{aligned}$$

Now we can transform $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ into cylindrical coordinates. To transform $\nabla \cdot \mathbf{F}$, we compute as follows.

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\
&= \left(\frac{\partial F_1}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial F_1}{\partial \theta}\frac{\partial \theta}{\partial x} \right) + \left(\frac{\partial F_2}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial y} \right) + \frac{\partial F_3}{\partial z} \\
&= \cos\theta \frac{\partial}{\partial \rho}\left(\cos\theta\ F_\rho - \sin\theta\ F_\theta\right) - \frac{1}{\rho}\sin\theta \frac{\partial}{\partial \theta}\left(\cos\theta\ F_\rho - \sin\theta\ F_\theta\right) \\
&+ \sin\theta \frac{\partial}{\partial \rho}\left(\sin\theta\ F_\rho + \cos\theta\ F_\theta\right) + \frac{1}{\rho}\cos\theta \frac{\partial}{\partial \theta}\left(\sin\theta\ F_\rho + \cos\theta\ F_\theta\right) + \frac{\partial F_z}{\partial z}
\end{aligned}
$$

After simplifying, we obtain

$$
\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}
$$

$\nabla \times \mathbf{F}$ is handled similarly.

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)\mathbf{k}
\end{aligned}
$$

Writing the partial derivatives of F_1, and F_2 in terms of F_ρ, F_θ, and their partial derivatives, we obtain

$$
\begin{aligned}
\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= \sin\theta \left(\frac{\partial F_z}{\partial \rho} - \frac{\partial F_\rho}{\partial z} \right) + \cos\theta \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \\
\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= \cos\theta \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) + \sin\theta \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \\
\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= \frac{1}{\rho}F_\theta + \frac{\partial F_\theta}{\partial \rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \theta}
\end{aligned}
$$

Writing $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in terms of $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\mathbf{e_z}$ and simplifying, we obtain

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \frac{1}{\rho}\left\{ \left(\frac{\partial F_z}{\partial \theta} - \rho\frac{\partial F_\theta}{\partial z} \right)\mathbf{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right)\rho\mathbf{e}_\theta + \left[\frac{\partial}{\partial \rho}\left(\rho F_\theta\right) - \frac{\partial F_\rho}{\partial \theta} \right]\mathbf{e_z} \right\} \\
&= \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}
\end{aligned}
$$