Math 20E
Stokes' Theorem

Theorem 1. *Let S be a C^2 (twice continuously differentiable) parametrized surface with C^1 (continuously differentiable) boundary ∂S and let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be C^1 on S and ∂S. Then,*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Proof. Let

$$\begin{array}{rcl} \mathbf{\Phi} : D \subset \mathbb{R}^2 & \longrightarrow & S \subset \mathbb{R}^3 \\ (u, v) & \longmapsto & \mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)) \end{array}$$

be a one-to-one positively-oriented C^2 parametrization of S, and let

$$\begin{array}{rcl} \mathbf{c} : [a, b] \subset \mathbb{R} & \longrightarrow & \partial D \subset \mathbb{R}^2 \\ t & \longmapsto & \mathbf{c}(t) = (u(t), v(t)) \end{array}$$

be a positively oriented C^1 parametrization of of ∂D. Then

$$\begin{array}{rcl} \mathbf{\Phi} \circ \mathbf{c} : [a, b] \subset \mathbb{R} & \longrightarrow & \partial S \subset \mathbb{R}^3 \\ t & \longmapsto & (\mathbf{\Phi} \circ \mathbf{c})(t) = \mathbf{\Phi}\left(u\left(t\right), v\left(t\right)\right) \end{array}$$

is a positively oriented C^1 parametrization of ∂S. Using the parametrization $\mathbf{\Phi}(u, v)$ for S, we can write

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D \big[(\nabla \times \mathbf{F})(\mathbf{\Phi})\big] \cdot \left(\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}\right) du\, dv$$

After a straightforward (but tedious) computation, one sees that

$$\big[(\nabla \times \mathbf{F})(\mathbf{\Phi})\big] \cdot \left(\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}\right) = \frac{\partial}{\partial u}\left(\mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial u}\right).$$

By Green's Theorem,

$$\iint_D \left[\frac{\partial}{\partial u}\left(\mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial u}\right)\right] du\, dv = \int_{\partial D} \mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial u}\, du + \mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial v}\, dv.$$

Thus,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial D} \mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial u}\, du + \mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial v}\, dv.$$

On the other hand,

$$\int_{\partial D} \mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial u}\, du + \mathbf{F}(\mathbf{\Phi}) \cdot \frac{\partial \mathbf{\Phi}}{\partial v}\, dv = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

since $\mathbf{\Phi} \circ \mathbf{c}(t) = \mathbf{\Phi}(u(t), v(t))$ is a parametrization of ∂S (because $\mathbf{\Phi}$ is a *one-to-one* parametrization of S). Stokes' Theorem follows. □