Math 142B
September 4, 2018

Question 1 For each index n, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^n} \text{ for some integer } k,\ 0 \le k \le 2^n \\ 0 & \text{otherwise} \end{cases}$$

Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$.

Then,

A. $\int_0^1 f_n = 0$ for every index n.

B. $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$.

C. $\int_0^1 f = 0$.

*D. **A** and **B**.

E. **A** and **C**.

Question 2 For each index n, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \le 1 \end{cases}$$

Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$. Then,

A. $\int_0^1 f_n = 1$ for every index n.

B. $\int_0^1 f = 0$.

C. $\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n$.

*D. **A** and **B**.

E. **A** and **C**.

Question 3 For each natural number n, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = e^x$. Then, $\{f_n\}$ converges pointwise on $[0,1]$ to f. We can also say that

A. $\{f_n\}$ converges uniformly to f on $[0,1]$.

B. $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$.

C. $\lim_{n\to\infty} f_n'(x) = f'(x)$ for each $x \in (0,1)$.

D. **B** and **C**.

*E. All of the above.

Question 4 Given a sequence of functions $\{f_n : [a, b] \to \mathbb{R}\}$ such that $\{f_n\}$ converges pointwise to f on $[a, b]$. Then we can say that

A. if f_n is integrable for every index n, then f is integrable.

B. if $\displaystyle\int_a^b f_n = 1$ for every index n, then $\displaystyle\int_a^b f = 1$.

C. if f_n is continuous for every index n, then f is continuous.

D. All of the above.

*E. None of the above.

Question 5 A sequence $\{a_k\}$ is Cauchy (or, is a Cauchy sequence) if for every $\varepsilon > 0$, there is an index N such that $|a_m - a_n| < \varepsilon$ for all indices $m, n \geq N$.
Given that $\{b_k\}$ is a Cauchy sequence. Then $\{b_k\}$ is

A. convergent.

B. bounded.

C. sequentially compact.

*D. **A** and **B**.

E. **B** and **C**.

Question 6 A sequence of functions $\{f_k\}$ is uniformly Cauchy on D if for every $\varepsilon > 0$, there is an index N such that $|f_m(x) - f_n(x)| < \varepsilon$ for all indices $m, n \geq N$ and all $x \in D$.
Given that $\{g_k\}$ is uniformly Cauchy on $[0, 1]$. Then,

A. For each $x \in [0, 1]$, $\{g_k(x)\}$ is a Cauchy sequence.

B. $\{g_k\}$ converges pointwise to a function $g : [0, 1] \to \mathbb{R}$.

C. $\{g_k\}$ converges uniformly to a function $g : [0, 1] \to \mathbb{R}$.

D. **A** and **B**.

*E. **A**, **B**, and **C**.