Math 142B
August 21, 2018

Question 1 The Mean Value Theorem asserts that if $f : [a, b] \to \mathbb{R}$ is continuous and its restriction $f : (a, b) \to \mathbb{R}$ is differentiable, then there is a point $x_0 \in (a, b)$ at which $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$.
The Mean Value Theorem can be proved by

*A. applying Rolle's Theorem to $g(x) = f(x) - f(a) - m(x - a)$ with $m := \frac{f(b)-f(a)}{b-a}$.

B. applying the Extreme Value Theorem to find a point $x_0 \in (a, b)$ at which $f'(x_0)$ has an extreme value.

C. applying the Mean Value Theorem for integrals to find x_0 so that $f(x_0) = \dfrac{1}{b - a} \displaystyle\int_a^b f$.

D. All of the above; **A**, **B**, and **C** are all part of the proof of the Mean Value Theorem.

E. None of the above; the Mean Value Theorem only applies to differentiable functions $f : [a, b] \to \mathbb{R}$.

Question 2 The Cauchy Mean Value Theorem asserts that if $f, g : [a, b] \to \mathbb{R}$ are continuous and their restrictions $f, g : (a, b) \to \mathbb{R}$ are differentiable with $g'(x) \neq 0$ for all $x \in (a, b)$, then there is a point $x_0 \in (a, b)$ at which $f'(x_0) = \frac{f(b)-f(a)}{g(b)-g(a)}$.
The Cauchy Mean Value Theorem can be proved by

*A. applying Rolle's Theorem to $h(x) = f(x) - mg(x)$ with the constant $m := \frac{f(b)-f(a)}{g(b)-g(a)}$.

B. applying the Mean Value Theorem to $h(x) = \frac{f(x)-f(a)}{g(x)-g(a)}$.

C. applying the Extreme Value Theorem to $h(x) = \frac{f(x)}{g(x)}$ in order to find a point x_0 at which $h'(x_0) = 0$.

D. all of the above; the Cauchy Mean Value Theorem can be proved in many ways.

E. none of the above; Cauchy had nothing to do with the Mean Value Theorem.

Question 3 Given a neighborhood I of x_0, n a nonnegative integer, and a function $f : I \to \mathbb{R}$ with $n+1$ derivatives such that $f^{(k)}(x_0) = 0$ for $0 \le k \le n$. Then for each $x \ne x_0$ in I, there is a c strictly between x and x_0 at which $f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$.
This statement

A. can be proven by applying the Cauchy Mean Value Theorem inductively to $f^{(k)}(x)$ and $g^{(k)}(x)$ for $0 \le k \le n$, starting with $f(x)$ and $g(x) = (x - x_0)^n$.

B. is a special case of the Lagrange Remainder Theorem since $p_n(x) = 0$ is the n^{th} Taylor polynomial for f at x_0.

C. immediately implies the Lagrange Remainder Theorem since $R_n(x) = f(x) - p_n(x)$ has contact of order n with the constant function 0.

D. **A** and **B**.

*E. **A**, **B**, and **C**.

Question 4 Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function $f(x) = e^x$. Then,

A. The n^{th} Taylor polynomial for f at $x = 0$ is
$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + \frac{1}{2} + \cdots + \frac{1}{n!} x^n.$$

B. $f^{(k)}(0) = p_n^{(k)}(0)$ for $0 \leq k \leq n$.

C. $\lim_{n \to \infty} p_n(x) = f(x)$ for every x.

D. **A** and **B**.

*E. **A**, **B**, and **C**.

Question 5 Let $H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ be the n^{th} harmonic number. Then,

A. $\lim_{n\to\infty} H_n = \infty$.

B. $\{c_n\}$, where $c_n = H_n - \log(n+1)$, is monotonically increasing and bounded above by 1.

C. $\lim_{n\to\infty} c_n = \gamma$, where $c_n = H_n - \log(n+1)$ and $\gamma \leq 1$ is called Euler's constant.

D. **B** and **C**.

*E. **A**, **B**, and **C**.

Question 6 Given $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ with derivatives of all orders. Then $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^m$ is the n^{th} Taylor polynomial for f at x_0, and

A. p_n has contact of order n with f at x_0.

B. $f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ for some c strictly between x and x_0.

C. $\lim_{n\to\infty} p_n(x) = f(x)$ for every x.

*D. **A** and **B**.

E. **A**, **B**, and **C**.