Math 142B
August 22, 2018

Question 1 Given a neighborhood I of x_0, n a nonnegative integer, and a function $f : I \to \mathbb{R}$ with $n+1$ derivatives such that $f^{(k)}(x_0) = 0$ for $0 \leq k \leq n$. Then for each $x \neq x_0$ in I, there is a c strictly between x and x_0 at which $f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$.
This statement

A. can be proven by applying the Cauchy Mean Value Theorem inductively to $f^{(k)}(x)$ and $g^{(k)}(x)$ for $0 \leq k \leq n$, starting with $f(x)$ and $g(x) = (x - x_0)^n$.

B. is a special case of the Lagrange Remainder Theorem since $p_n(x) = 0$ is the n^{th} Taylor polynomial for f at x_0.

C. immediately implies the Lagrange Remainder Theorem since $R_n(x) = f(x) - p_n(x)$ has contact of order n with the constant function 0.

D. **A** and **B**.

*E. **A**, **B**, and **C**.

Question 2 Let $H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ be the n^{th} harmonic number, and let $c_n = H_n - \log(n+1)$. Then,

A. $c_n > 0$ for every index n.

B. $\{c_n\}$ is monotonically increasing and bounded above by 1.

C. $\lim_{n\to\infty} c_n = \gamma$, where $\gamma \leq 1$ is called Euler's constant.

D. **B** and **C**.

*E. **A**, **B**, and **C**.

Question 3 Given a neighborhood I of a point x_0 and an infinitely differentiable function $f : I \to \mathbb{R}$. Then, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ for every $x \in I$ whenever

A. $\lim_{n\to\infty} [f(x) - p_n(x)] \to 0$ for every $x \in I$, where p_n is the n^{th} Taylor polynomial for f at x_0.

B. There is an $M > 0$ for which $\left|f^{(k)}(x)\right| \leq M^k$ for every $x \in I$ and every index k.

C. $\lim_{n\to\infty} \frac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)^{n+1} = 0$ for every $x \in I$.

*D. **A** and **B**.

E. **A**, **B**, and **C**.

[Note: Whether or not **C** is true is a Math 142B open question.]

Question 4 $p_n(x) = \sum_{k=0}^{n} x^k$ is the n^{th} Taylor polynomial at $x = 0$ for $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{1-x}$. Moreover, $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ for every index n. We can say that $f(x) = \sum_{k=0}^{\infty} x^k$ for all $x \in (-1, 1)$ because

*A. $\lim_{n\to\infty} [f(x) - p_n(x)] = 0$ for every $x \in (-1, 1)$.

B. There is some $M > 0$ for which $\left|f^{(n)}(x)\right| \leq M^n$ for all $x \in (-1, 1)$.

C. $\lim_{n\to\infty} \dfrac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)^{n+1} = 0$ for every $x \in (-1, 1)$.

D. All of the above.

E. None of the above.

[Note: Since $\lim_{x\to 1^-} f^{(k)}(x) = +\infty$ for every index k, **B** and **C** need to be stated more precisely before we can conclude that they're true.]

Question 5 Given a function $f : [a, b] \to \mathbb{R}$. We can say that

A. if f is bounded, then f must have both a maximum and a minimum.

B. if f is bounded, then f must have either a maximum or a minimum.

C. if f is a constant function, then f has neither a maximum nor a minimum.

*D. f need not attain a maximum or a minimum.

E. **C** and **D**.

Question 6 Given a function $f : [a, b] \to \mathbb{R}$. Then,

A. if f is bounded, then f is continuous.

B. if f is monotonically increasing then f is continuous.

C. if f is not continuous, then f is not integrable.

D. if f is integrable, then f is continuous.

*E. None of the above: all the above statements are false.