Math 142B
August 28, 2018

Question 1 Given a continuous function $f : [0,1] \to \mathbb{R}$. Given $\varepsilon > 0$, there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$. We can construct such a polynomial p by defining

*A. $p(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$ for all x and an appropriately chosen index n.

B. $p(x) := \sum_{k=0}^{n} \frac{f^{(k)}\left(\frac{1}{2}\right)}{k!}\left(x - \frac{1}{2}\right)^k$ for all x and an appropriately chosen index n.

C. $p(x) := \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$ for all x and an appropriately chosen index n.

D. **A** and **B**; both methods produce a polynomial p with the required property.

E. **A**, **B**, and **C**; all three methods produce a polynomial p with the required property.

Question 2 Given a continuous function $f : [a, b] \to \mathbb{R}$. Define $g : [0, 1] \to [a, b]$ by $g(t) = a + (b - a)t$. Then,

A. $f \circ g : [0, 1] \to \mathbb{R}$ given by $(f \circ g)(t) = f(g(t))$ is continuous.

B. Given $\varepsilon > 0$, there is a polynomial q such that $|(f \circ g)(t) - q(t)| < \varepsilon$ for all $t \in [0, 1]$.

C. Given $\varepsilon > 0$ and q as in **B**, set $p(x) = q\left(\frac{x-a}{b-a}\right)$. Then, $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$

*D. All of the above.

E. None of the above.

Question 3 Given any number β. Define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = (1 + x)^{\beta}$. Then,

*A. $f^{(k)}(x) = \beta(\beta - 1) \cdots (\beta - k + 1)(1 + x)^{\beta - k}$.

B. $f(x) = \sum_{k=0}^{\infty} \frac{\beta(\beta - 1) \cdots (\beta - k + 1)}{k!}(1 + x)^{\beta - k}$.

C. $f(x) = \sum_{k=0}^{\infty} \binom{\beta}{k}(1 + x)^{\beta - k}$.

D. **B** and **C**; they are the same since $\binom{\beta}{k} = \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$.

E. All of the above.

Question 4 Given a neighborhood I of x_0 and an infinitely differentiable function $f : I \to \mathbb{R}$. Then,

A. $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(x_0)(x-t)^n \, dt,$ for every $x \in I$ and every index n.

B. For every $x \in I$ and every index n, there is a c strictly between x and x_0 at which
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

C. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ for every $x \in I$ and every index n.

D. All of the above.

*E. **A** and **B**.

Question 5 The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \end{cases}$$ is an example of a function that is

A. bounded.

B. infinitely differentiable.

C. analytic.

D. All of the above.

*E. **A** and **B**.

Question 6 Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function $f(x) = e^x$. Then,

A. The n^{th} Taylor polynomial for f at $x = 0$ is
$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + \frac{1}{2} + \cdots + \frac{1}{n!}x^n.$$

B. $f^{(k)}(0) = p_n^{(k)}(0)$ for $0 \leq k \leq n$.

C. $\lim_{n\to\infty} p_n(x) = f(x)$ for every x.

D. **A** and **B**.

*E. **A**, **B**, and **C**.