Math 142B
August 29, 2018

Question 1 Given a neighborhood I of a point x_0 and an infinitely differentiable function $f : I \to \mathbb{R}$. Then, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ for every $x \in I$ whenever

A. $\lim_{n\to\infty} [f(x) - p_n(x)] \to 0$ for every $x \in I$, where p_n is the n^{th} Taylor polynomial for f at x_0.

B. There is an $M > 0$ for which $\left|f^{(k)}(x)\right| \leq M^k$ for every $x \in I$ and every index k.

C. $\lim_{n\to\infty} \frac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)^{n+1} = 0$ for every $x \in I$.

*D. **A** and **B**.

E. **A**, **B** and **C**.

[Note: Whether or not **C** is true is a Math 142B open question.]

Question 2 Given any number β. Define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = (1 + x)^{\beta}$. Then,

A. $f^{(k)}(x) = \beta(\beta - 1) \cdots (\beta - k + 1)(1 + x)^{\beta - k}$, thus $f^{(k)}(0) = \beta(\beta - 1) \cdots (\beta - k + 1)$.

B. $f(x) = \sum_{k=0}^{\infty} \frac{\beta(\beta - 1) \cdots (\beta - k + 1)}{k!} x^k$.

C. $f(x) = \sum_{k=0}^{\infty} \binom{\beta}{k} x^k$.

D. **B** and **C**; they are the same since $\binom{\beta}{k} = \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$.

*E. All of the above.

Question 3 For each index n, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^n} \text{ for some integer } k,\ 0 \le k \le 2^n \\ 0 & \text{otherwise} \end{cases}$$

Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$.

Then,

A. $\int_0^1 f_n = 0$ for every index n.

B. $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$.

C. $\int_0^1 f = 0$.

*D. **A** and **B**.

E. **A** and **C**.

Question 4 For each index n, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \le 1 \end{cases}$$

Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$.

Then,

A. $\int_0^1 f_n = 1$ for every index n.

B. $\int_0^1 f = 0$.

C. $\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n$.

*D. **A** and **B**.

E. **A** and **C**.

Question 5 Given a sequence of functions $\{f_n : [a, b] \to \mathbb{R}\}$ such that $\{f_n\}$ converges pointwise to f on $[a, b]$. Then we can say that

A. if f_n is integrable for every index n, then f is integrable.

B. if $\int_a^b f_n = 1$ for every index n, then $\int_a^b f = 1$.

C. if f_n is continuous for every index n, then f is continuous.

D. All of the above.

*E. None of the above.

Question 6 Given a neighborhood I of a point x_0 and a function $f : I \to \mathbb{R}$ with $n+1$ derivatives. Let $p_n(x)$ be the n^{th} Taylor polynomial for f at $x = x_0$. Then,

A. There is a c strictly between x and x_0 at which
$$f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

B. There is a c strictly between x and x_0 at which
$$\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt.$$

C. $$\lim_{n\to\infty}\left\{\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt\right\} = 0.$$

*D. **A** and **B**.

E. **A**, **B**, and **C**.