Math 142B **Homework 4** **Due August 30, 2018**

1. Let $H_n := \sum_{k=1}^{n} \frac{1}{k} = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ be the n^{th} harmonic number, and define $c_n := H_n - \log(n+1)$.
   (a) Prove that $0 < x - \log(1+x) < \frac{1}{2}x^2$ for all $x > 0$.
   (b) Prove that $\{c_n\}$ is monotonically increasing.
   (c) Prove that $c_n < 1$ for every index n. Conclude that $\lim_{n\to\infty} c_n = \gamma$ for some $\gamma \leq 1$. [γ is called Euler's constant.]

2. Let $f : (0,2) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$.
   (a) Find p_n, then n^{th} Taylor polynomial for f at $x_0 = 1$.
   (b) Show that for every natural number n, $f(x) - p_n(x) = \dfrac{(1-x)^{n+1}}{x}$ if $0 < x < 2$.
   (c) Prove that $f(x) = \sum_{k=0}^{\infty} \dfrac{f^{(k)}(1)}{k!}(x-1)^k$ if $|x-1| < 1$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and that

$$\begin{cases} f''(x) - f'(x) - f(x) = 0 & \text{for all } x \\ f(0) = 1 \quad \text{and} \quad f'(0) = 1. \end{cases}$$

   (a) Find a recursive formula for the coefficients of the n^{th} Taylor polynomial for f at $x = 0$.
   (b) Show that the Taylor expansion converges at every point.

4. Explain how the identity

$$s = \frac{1 + \left(\frac{s-1}{s+1}\right)}{1 - \left(\frac{s-1}{s+1}\right)} \quad \text{if} \quad s \neq 0$$

allows one to efficiently compute the value of $\log(1+x)$ if $0 < x < 1$ and x is close to 1.

5. Given continuous functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ with $g(x) \geq 0$ for all $x \in [a,b]$. Show that there is a point $c \in (a,b)$ at which

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

6. By applying the Cauchy Integral Remainder Theorem, we obtain the formula

$$\log(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}x^k + (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt \quad \text{for all} \quad x > -1 \quad \text{and every index} \quad n.$$

Show that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}x^k \quad \text{for all} \quad x \in (-1,1].$$

[Note: As we'll see later, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}x^k$ diverges when $x > 1$.]

7. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ e^{-\frac{1}{x^2}} & \text{if } x \neq 0. \end{cases}$$

Show that, given $r > 0$, there is no $M > 0$ such that for every index n, $\left|f^{(n)}(x)\right| \leq M^n$ for all $x \in [-r, r]$.

8. Let n and k be indices with $1 \leq k \leq n$.
   (a) Show that $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$.
   (b) Use the result of part (a) and equation (8.42) from your textbook to verify that $\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} = x$.

9. Let n and k be indices with $2 \leq k \leq n$.
   (a) Show that $\frac{k(k-1)}{n(n-1)}\binom{n}{k} = \binom{n-2}{k-2}$.
   (b) Use the result of part (a) and equation (8.42) from your textbook to verify that $\sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)}\binom{n}{k} x^k (1-x)^{n-k} = x^2$.

10. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \left|x - \frac{1}{2}\right|$. Using the proof of the Weierstrass Approximation Theorem, find an explicit formula for a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - p(x)| < \frac{1}{4}$ for all $x \in [0, 1]$.