Math 142B
Final Examination
September 5, 2014
...

Instructions

1. **You may use any type of calculator, but no other electronic devices during this exam.**
2. **You may use one page of notes, but no books or other assistance during this exam.**
3. **Write your *Name*, *PID*, and *Section* on the front of your Blue Book.**
4. **Write your solutions clearly in your Blue Book**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order they appear in the exam.**
   (c) **Start each question on a new side of a page.**
5. **Read each question carefully, and answer each question completely.**
6. **Show all of your work; no credit will be given for unsupported answers.**

1. Given $f : [0,1] \to \mathbb{R}$ a continuous function. Show that

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right] = \int_0^1 f.$$

2. Given $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ x+1 & \text{if } 1 < x \le 2. \end{cases}$$

   (a) Determine the mean value of f.
   (b) Show that $f(x)$ is not equal to the mean value of f for any $x \in [0,2]$. Explain why this does not contradict the Mean Value Theorem.

3. Expand the polynomial $p(x) = (x+1)^5 - (x+1)^3 + (x+1)$ in powers of x.

4. Given $f : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable. Use the Second Fundamental Theorem to show that

$$f(x) = f(0) + f'(0)\,x + \int_0^x (x-t)f''(t)\,dt \quad \text{for all } x.$$

(Remark: $\int_0^x (x-t)f''(t)\,dt$ is the Cauchy integral remainder for the first Taylor polynomial of f centered at $x_0 = 0$.)

Exam continues on other side.

5. Consider the function $f : (0, 2) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.

   (a) Find a formula for p_n, the n^{th} Taylor polynomial for f centered at $x_0 = 1$.

   (b) Use the Geometric Sum Formula to show that
   $$f(x) - p_n(x) = \frac{(1-x)^{n+1}}{x} \quad \text{for every natural number } n.$$

   (c) Prove that
   $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k \quad \text{for all } |x-1| < 1.$$
   That is, prove that the Taylor series for $f(x)$ centered at $x_0 = 1$ converges for all $|x - 1| < 1$.

6. Consider the sequence of functions $\{f_n : [0, 1] \to \mathbb{R}\}$ defined by $f_n(x) = 2nx\, e^{-nx^2}$ for each $n \in \mathbb{N}$.

   (a) Determine the limit function $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

   (b) Show that $\displaystyle\int_0^1 f \neq \lim_{n\to\infty} \int_0^1 f_n$.

   (c) Does f_n converge uniformly to f? Briefly justify your answer.

7. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) Find a sequence $\{P_n\}$ of partitions of $[0, 1]$ that form an Archimedean sequence of partitions for f and verify that $\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$.
   $\big[$Hint: Write $P_n = \{x_0, \dots, x_{k_n+1}\}$, set $x_1 = \frac{1}{n}$, and set $x_i - x_{i-1} = \frac{1}{k_n}\left(1 - \frac{1}{n}\right)$ for $2 \le i \le k_n + 1$ and sufficiently large k_n.$\big]$

   (b) Determine the value of $\displaystyle\int_0^1 f$.