An Elementary Introduction to Juggling and Juggling Mathematics

- A description of juggling

- Mathematical notation for simple juggling patterns

- Demonstration of some simple juggling patterns

- Brief introduction to mathematics of juggling

References

1. Polster, Burkhard. *The Mathematics of Juggling*. Springer-Verlag, New York, 2002

2. Knutson, Allen. *Siteswap FAQ*. http://www.juggling.org/help/siteswap/

3. Buhler, Joe; Eisenbud, David; Graham, Ronald; Wright, Colin. *Juggling drops and descents*. http://www.juggling.org/papers/

4. Beek, Peter; Lewbel, Arthur. *The science of juggling*. http://www.juggling.org/papers/

5. Hall, Marshall *A Combinatorial Problem on Abelian Groups*, Proceedings of the AMS 3, 1952

Resources

1. Juggling Lab
   http://www.jugglinglab.sourceforge.net/

2. Internet Juggling Database
   http://www.jugglingdb.com/

3. Juggling Information Service
   http://www.juggling.org/

Simple Juggling Patterns

1. Balls are thrown and caught at equally spaced moments in time (beats).

2. Juggling patterns are periodic.

3.(a) At most one ball gets caught and thrown on every beat.

(b) When a ball is caught, the same ball is thrown.

4. One hand throws on odd-numbered beats; the other throws on even-numbered beats.

Note: Condition 3 distinguishes *simple* patterns from *multiplex* patterns.

Describing Simple Juggling Patterns

1. Number the beats; at most one ball will be thrown on each beat.

2. Record the number of beats the thrown ball is in the air before being caught (the "height" of the throw).

3. Write the finite sequence of heights to describe the pattern.

Example

beats:	...	-3	-2	-1	0	1	2	3	...
heights:	...	5	1	5	1	5	1	5	...

Pattern: 51 (Heights assumed less than 10)

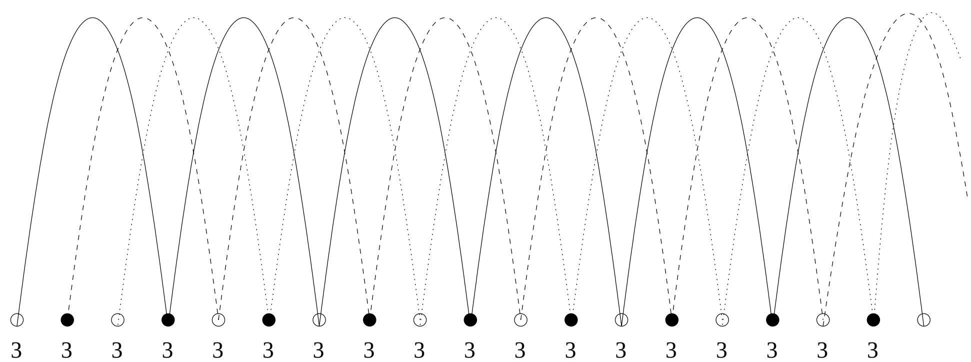
Juggling Diagrams

1. Indicate the beats by dots.

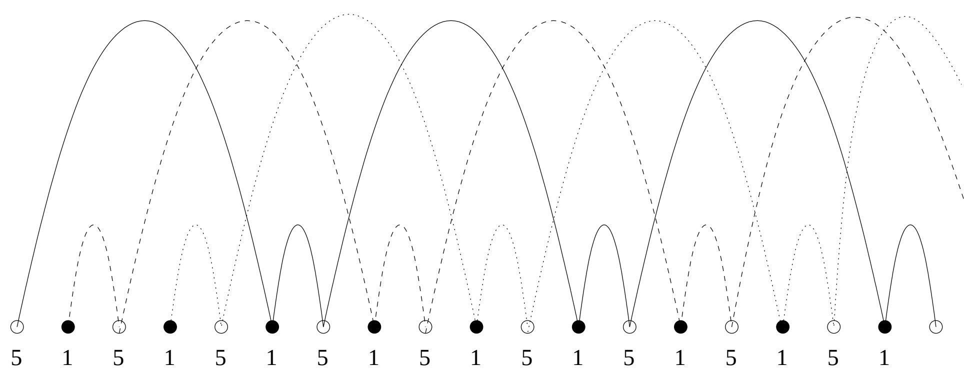
2. Indicate the throws by curves connecting the dots; the difference in beats of the curves endpoints (beginning beat and ending beat) is the throw's height.

Note: Two (or more) balls cannot land on the same beat (a collision); for example, 321 (or any sequence containing adjacent terms of the form $n, (n-1)$ is not a juggling pattern.

Examples



3: 3-Ball Cascade



51: 3-Ball Shower

Examples

p^* means I can't juggle the pattern p — yet. Hey! I'm just an amateur!

0-ball pattern: 0

1-ball patterns: 1, 20

2-ball patterns: 2, 31, 312, 330, 411, 40, 501

3-ball patterns: 3, 51, 423, 52512, 441, 4413*, 531*, 50505*, 504*

4-ball patterns: 4, 552, 53*, 534*, 5551*, 55550*

5-ball patterns: 5, 64*, 663*, 771*, 7571*

Definition. A *juggling sequence* is a sequence (a_i) of nonnegative integers such that $i \mapsto i + a_i$ is a permutation of $\mathbb{Z}$.

Note:

1. By this definition, a juggling sequence need not be periodic. (However, we will focus on periodic juggling sequences.)

2. The condition on (a_i) is equivalent to the existence of a juggling diagram for (a_i).

Questions

1. How many balls are required to juggle a given periodic juggling sequence?

2. When is a periodic sequence a juggling sequence?

Definition. Let (a_i) be a juggling sequence.

1. $\text{height}(a_i) := \sup_{i \in \mathbb{Z}} a_i$

2. $\text{balls}(a_i) := \#\{\text{orbits in juggling diagram of } a_i\}$

Theorem. Let (a_i) be a juggling sequence.

1. If height(a_i) is finite, then

   $$\lim_{|I|\to\infty} \frac{\sum_{i\in I} a_i}{|I|}$$

   is finite and equal to balls(a_i), where the limit is over all integer intervals

   $$I = \{c, c+1, c+2, \ldots, d\} \subset \mathbb{Z}$$

   and $|I| = d - c + 1$, the number of integers in I.

2. The number of balls required to juggle a periodic juggling sequence is equal to its average.

Corollary. (Average Test) If the average of a finite sequence of nonnegative integers is not an integer, then the sequence is not a juggling sequence.

Note:

1. This provides a simple necessary condition for a periodic integer sequence to be a juggling sequence.

2. We have used the convention that a finite sequence corresponds to a periodic sequence by repetition.

The following theorem provides a (slightly less) simple necessary and sufficient condition for a periodic integer sequence to be a juggling sequence.

Theorem. (Permutation Test) Let $s = \{a_i\}_{i=0}^{p-1}$ be a sequence of nonnegative integers and let $[p] = \{0, 1, 2, \ldots, p-1\}$. Then, s is a juggling sequence if and only if the function

$$\begin{aligned} \phi_s : [p] &\rightarrow [p] \\ i &\mapsto (i + a_i) \mod p \end{aligned}$$

is a permutation of $[p]$.

Example: $s = \{3, 1, 2\}$

$$\begin{aligned} \phi_s : 0 &\mapsto (0 + 3) \mod 3 = 0 \\ 1 &\mapsto (1 + 1) \mod 3 = 2 \\ 2 &\mapsto (2 + 2) \mod 3 = 1 \end{aligned}$$

Theorem. ("Converse" of Average Test) Given a finite sequence of nonnegative integers whose average is an integer, there is a permutation of the sequence that is a juggling sequence.

Example: 321 is not a juggling sequence, but 312 is a juggling sequence.

This result is based on a theorem about abelian groups proved by Marshall Hall in 1952

[*A Combinatorial Problem on Abelian Groups*, Proceedings of the AMS 3 (1952), pg 584-587].

There's more! For example, we could ask the following questions.

1. How many juggling sequences of period p are there with at most b balls?

2. How many juggling sequences of period p are there with exactly b balls?

3. How many juggling sequences of period p are there with height at most h?

Be well, have fun, and juggle!