Math 102 Homework Assignment 2
Due Thursday, January 20, 2022

1. Let L be a linear operator on $\mathbb{R}^n$ with the property that $L(\mathbf{x}) = \mathbf{0}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$. Let $A = \big[L\big]_{\mathcal{E}}$ be the matrix representing L with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. Show that A is singular.

2. Let L be a linear operaton on a vector space V. Let $A = \big[L\big]_{\mathcal{B}}$ be the matrix representing L with respect to a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V. Show that A^m is the matrix representing L^m with respect to $\mathcal{B}$; that is, show that $\big[L^m\big]_{\mathcal{B}} = \Big(\big[L\big]_{\mathcal{B}}\Big)^m$.

3. Suppose $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. Write $S = \begin{bmatrix}\mathbf{s}_1 & \cdots & \mathbf{s}_n\end{bmatrix}$; that is, $\mathbf{s}_i$ is the i^{th} column of S.
   (a) Show that $A\mathbf{s}_i = \lambda_i \mathbf{s}_i$ for $i = 1, \ldots, n$.
   (b) Show that if $\mathbf{x} = \alpha_1\mathbf{s}_1 + \alpha_2\mathbf{s}_2 + \cdots + \alpha_n\mathbf{s}_n$, then $A^k\mathbf{x} = \alpha_1\lambda_1^k\mathbf{s}_1 + \alpha_2\lambda_2^k\mathbf{s}_2 + \cdots + \alpha_n\lambda_n^k\mathbf{s}_n$.
   (c) Suppose $|\lambda_i| < 1$ for $i = 1, \ldots, n$. What happens to $A^k\mathbf{x}$ as $k \to \infty$? Explain.

4. Let A and B be similar matrices.
   (a) Show that A^T and B^T are similar.
   (b) Show that A^k and B^k are similar for every positive integer k.

5. A linear transformation $L : V \to W$ is said to be *one-to-one* if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$.
   (a) Show that if $\ker(L) = \{\mathbf{0}_V\}$, then L is one-to-one.
   (b) Show that if L is one-to-one, then $\ker(L) = \{\mathbf{0}_V\}$.

6. Let V be the subspace of $C[a, b]$ spanned by $\{1, e^x, e^{-x}\}$, and let D be the differentiation operator on V.
   (a) Find the transition matrix P representing the change of coordinates from the ordered basis $E = \{1, e^x, e^{-x}\}$ to the ordered basis $H = \{1, \sinh(x), \cosh(x)\}$.
   (b) Find the matrix $A = [D]_H$ representing D with respect to the basis H.
   (c) Find the matrix $B = [D]_E$ representing D with respect to the basis E.
   (d) Verify that $B = P^{-1}\,A\,P$.