Math 102 Homework Assignment 3
Due Thursday, February 3, 2022

1. Let A be a $m \times n$ matrix. Show that:
   (a) If $\mathbf{x} \in N\left(A^T A\right)$, then $A\mathbf{x} \in R(A) \cap N\left(A^T\right)$.
   (b) $N\left(A^T A\right) = N(A)$.

2. Let V and W be subspaces of $\mathbb{R}^n$ such that $V \subset W$. Show that $W^\perp \subset V^\perp$.

3. Suppose A is a symmetric $n \times n$ matrix. Let V be a subspace of $\mathbb{R}^n$ with the property that $A\mathbf{x} \in V$ for every $\mathbf{x} \in V$. Show that $A\mathbf{y} \in V^\perp$ for every $\mathbf{y} \in V^\perp$. (Remark: The subspace V is said to be invariant under A. This exercise shows that if V is an invariant subspace under A, then so is $V^\perp$.)

4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$.
   (a) Find the orthogonal projection of $\mathbf{b}$ onto $R(A)$.
   (b) Describe all least squares solutions to $A\mathbf{x} = \mathbf{b}$.

5. Let A be a $n \times n$ matrix. Given that $\begin{pmatrix} A & I \\ O & A^T \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}$.
   Show that $\hat{\mathbf{x}}$ is a least squares solution of the system $A\mathbf{x} = \mathbf{b}$ and that $\mathbf{r}$ is the residual vector.

6. Given a $m \times n$ matrix A. Let $\hat{\mathbf{x}}$ be a solution to the least squares problem $A\mathbf{x} = \mathbf{b}$. Show that a vector $\mathbf{y} \in \mathbf{R}^n$ will also be a least squares solution if an only if $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$ for some vector $\mathbf{z} \in N(A)$.

7. Let P_3 be the inner product space of polynomials of degree less than 3 with inner product $\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)\,dt$. Let $g_1(t) = 1$ and $g_2(t) = t$. Find a basis for $\text{Span}(g_1, g_2)^\perp$, the orthogonal complement of the subspace of P_3 spanned by g_1 and g_2.

8. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in an inner product space V.
   (a) Show that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.
   (b) Show that if $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.