Math 102 Homework Assignment 4
Due Thursday, February 10, 2022

1. Let $A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
   (a) Find the projection matrix P that projects vectors in $\mathbb{R}^4$ onto $R(A)$.
   (b) Find an orthonormal basis for $N(A^T)$.
   (c) Determine the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $N(A^T)$.

2. Let A be a $m \times n$ matrix, let P be the projection matrix that projects vectors in $\mathbb{R}^m$ onto $R(A)$, and let Q be the projection matrix that projects vectors in $\mathbb{R}^n$ onto $R(A^T)$.
   (a) Show that $I - P$ is the projection matrix from $\mathbb{R}^m$ onto $N(A^T)$.
   (b) Show that $I - Q$ is the projection matrix from $\mathbb{R}^n$ onto $N(A)$.

3. Let $\mathbf{v}$ be a vector in an inner product space V and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto an n-dimensional subspace S of V. Show that $\|\mathbf{p}\|^2 = \langle \mathbf{p}, \mathbf{v} \rangle$.

4. Consider the inner product space $C[0,1]$ of continuous functions on $[0,1]$ with inner product
$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$
   Find an orthonormal basis for $\mathrm{Span}(1, x, x^2)$, the subspace of $C[0,1]$ spanned by $\{1,\ x,\ x^2\}$.

5. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k,\ \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ be an orthonormal basis for an inner product space V. Let $S_1 = \mathrm{Span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, the subspace of V spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ and $S_2 = \mathrm{Span}(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n)$, the subspace of V spanned by $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$. Show that $S_1 \perp S_2$.

6. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and define $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that:
   (a) H is symmetric.
   (b) H is orthogonal.
   (c) $H^{-1} = H$.

7. (a) Given two $n \times n$ orthogonal matrices Q_1 and Q_2. Show that $Q_1 Q_2$ is an orthogonal matrix.
   (b) Given two $n \times n$ permutation matrices P_1 and P_2. Is $P_1 P_2$ a permutation matrix? Why or why not?

8. Let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$ be an $n \times n$ orthogonal matrix.
   Show that $\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T = I$, the $n \times n$ identity matrix.