1. Suppose someone believed that, if they looked at large enough numbers, they could find an integer that was neither even nor odd. How could you convince them that this is not possible?

2. Bob has 10 pockets and 44 silver dollars. He wants to put his dollars into his pockets distributed so that each pocket contains a different number of dollars. Can he do this? Why or why not?

3. Are there any Pythagorean triples (integers $a$, $b$, and $c$ such that $a^2 + b^2 = c^2$) consisting of three odd numbers? Justify your answer.

4. Sarah says that there are essentially the same number of even integers as integers. What might be her reasoning?

5. Mike’s eccentric boss gives him a choice of salary: after working for $n$ days, he can either be paid $n^2$ or $2^n$ dollars. Which should he choose? Does it depend on how many days he will work for? Prove your answer.

6. For what values of $n$ is $3^n < n!$? Prove your answer.

7. Rani walks into a bank wanting to make a withdrawal. The teller says: “The only bills we carry at this bank are $2$ bills and $5$ bills. Nevertheless, as long as you want to withdraw a whole number bigger than $3$, I can help you.” Prove the teller’s claim that any amount of money over $3$ can be paid using only $2$ bills and $5$ bills.

8. Prove that if any line of people begins with a woman and ends with a man, then somewhere in the line a man must be standing directly behind a woman.

9. Are there any equalities (like $1 + 2 + \ldots + n = n(n + 1)/2$) that are true for the first million values of $n$, and then false after that? If so, can you find one? If not, why not?

10. Prove that $x$ is irrational if and only if $10x$ is irrational.

11. Prove that for all positive integers $n$, $\log(a_1 \cdot a_2 \cdots a_n) = \log a_1 + \log a_2 + \cdots + \log a_n$.

12. Think of or find at least 3 definitions for an infinite set. Which one would you prefer to use to test whether some given set was infinite and why? Which one would you prefer in proving some fact about a set that is assumed to be infinite and why?

13. Find an upper bound to the sequence: \[
\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots
\]
Explain how you know your answer is an upper bound.
14. You are given $3^n$ coins, all identical except for one which is heavier. Prove that you can find the heavy coin in $n$ weighings with a balance.

15. Let $n$ be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped pieces, where these pieces cover three squares at a time.

16. The Tower of Hanoi Problem: Three pegs are stuck in a board. On one of these pegs is a pile of disks graduated in size, the smallest being on top. The object of this puzzle is to transfer the pile to one of the other two pegs by moving the disks one at a time from one peg to another in such a way that a disk is never placed on top of smaller disk. How many moves are needed to transfer a pile of $n$ disks?

17. Prove that every third Fibonacci number is even.

18. Into how many regions is the plane cut by $n$ lines, assuming no two lines are parallel and no three intersect at a point. Prove your answer.

19. Find and prove a formula for the number of subsets of the set \{1, 2, \ldots, n\}.

20. Prove that for any positive integer $n \geq 4$, $2^n \leq n!$.

21. Write $\mathbb{R}$ as a disjoint union of countably infinite sets. (Make sure to carefully show that the sets are disjoint, countable, and cover all of $\mathbb{R}$).