

Math 109
HW 2 Graded Solutions
Summer, 2010

Problem II.13 Prove that, for sets A , B , C , and D .

i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Pf)

\supseteq) Suppose $(x_0, y_0) \in (A \times B) \cup (A \times C)$.

Then, $(x_0, y_0) \in (A \times B)$ or $(x_0, y_0) \in (A \times C)$

$$\begin{aligned} &\Rightarrow (x_0 \in A \text{ and } y_0 \in B) \text{ or } (x_0 \in A \text{ and } y_0 \in C) \\ &\Rightarrow x_0 \in A \text{ and } (y_0 \in B \text{ or } y_0 \in C) && \text{(by distributivity)} \\ &\Rightarrow x_0 \in A \text{ and } y_0 \in B \cup C \\ &\Rightarrow (x_0, y_0) \in A \times (B \cup C) \end{aligned}$$

\subseteq) Suppose $(x_0, y_0) \in A \times (B \cup C)$.

Then, $x_0 \in A$ and $y_0 \in B \cup C$

$$\begin{aligned} &\Rightarrow x_0 \in A \text{ and } (y_0 \in B \text{ or } y_0 \in C) \\ &\Rightarrow (x_0 \in A \text{ and } y_0 \in B) \text{ or } (x_0 \in A \text{ and } y_0 \in C) && \text{(by distributivity)} \\ &\Rightarrow (x_0, y_0) \in (A \times B) \text{ or } (x_0, y_0) \in (A \times C) \\ &\Rightarrow (x_0, y_0) \in (A \times B) \cup (A \times C) \end{aligned}$$

$\therefore A \times (B \cup C) \supseteq (A \times B) \cup (A \times C)$ and $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$,
therefore $A \times (B \cup C) = (A \times B) \cup (A \times C)$ is indeed true.

ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Pf:

$$\begin{aligned} (x, y) \in (A \cap C) \times (B \cap D) &\Leftrightarrow (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D) \\ &\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in D) \\ &\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D) \end{aligned}$$

Thus, $(x, y) \in (A \cap C) \times (B \cap D) \Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D)$ as required. \square

Problem IV.4 Prove there is no rational number whose square is 98.

Proof 1:

Suppose for contradiction there is a rational number $\frac{a}{b}$, where $a, b \in \mathbb{Z}$, $b \neq 0$, whose square is 98. Then $(\frac{a}{b})^2 = 98 \Rightarrow a^2 = 98b^2 \Rightarrow \frac{a^2}{2} = 49b^2 \Rightarrow \frac{a^2}{49b^2} = 2$. So $\frac{a}{7b} = \sqrt{2}$. Let $c \in \mathbb{Q}$ such that $c = \frac{a}{7b}$. Thus $c = \sqrt{2}$, but $\sqrt{2}$ is an irrational number, so we have a contradiction. Hence there is no rational number whose square is 98. ■

Proof 2:

Suppose there is a number q such that $q^2 = 98$. Write q as a fraction in lowest terms, $q = \frac{a}{b}$, so that a and b are integers and $\gcd(a, b) = 1$. Now $q^2 = 98$, thus $\frac{a^2}{b^2} = 98$ and $a^2 = 98b^2$. Hence, $a^2 = 2(7b)^2$. Thus, a^2 is divisible by 2, so it is even. It follows that a is also even, and can be written as $a = 2a_1$. Hence $4a_1^2 = 2(7b)^2$, and $(7b)^2 = 2a_1^2$. Thus, $(7b)^2$ is also even, and $7b$ is even. If b is odd, $7b$ is odd, so b must be even. Thus, $b = 2b_1$. Now $q = \frac{2a_1}{2b_1}$, so a and b are not in lowest terms. Thus, there are no two integers a and b such that $q = \frac{a}{b}$ in lowest terms, and it follows that q is not a rational number.

Proof 3:

Assume, for the sake of contradiction that the rational number $\frac{p}{q}$ squared is equal to 98. Here, p and q are both integers. We will look at the pair (p, q) such that the quantity $|p| + |q|$ is minimized. So we see that

$$\frac{p^2}{q^2} = 98 \Leftrightarrow p^2 = 98q^2.$$

Notice that the right hand side of the equation is divisible by 2. Therefore the left hand side must be divisible by 2. Since p^2 is even, $p = 2r$ for some integer r . Now the equation becomes:

$$\begin{aligned}(2r)^2 &= 98q^2 \\ 4r^2 &= 98q^2 \\ 2r^2 &= 49q^2\end{aligned}$$

Notice that the left hand side of the equation is divisible by 2. Since $2 \nmid 49$, q^2 must be divisible by 2, so q is even. So let $q = 2s$ for some integer s . The equation becomes:

$$\begin{aligned}2r^2 &= 49(2s)^2 \\ 2r^2 &= 196s^2 \\ r^2 &= 98s^2 \\ \frac{r^2}{s^2} &= 98\end{aligned}$$

Thus the square of the rational number $\frac{r}{s}$ is also 98. However, $|r| + |s| = |\frac{p}{2}| + |\frac{q}{2}| = \frac{|p| + |q|}{2} < |p| + |q|$ contradicting the minimality of the choice of (p, q) .