1. Let \( S \subset \mathbb{C} \) be the set \( S = \{ z = -it \mid t \geq 0 \} \). Define explicitly a continuous branch of \( \log(z) \) on \( \mathbb{C} \setminus S \); that is, the complex plane slit along the negative imaginary axis. (Hint: Recall that \( \log(z) \), the principal branch of \( \log(z) \), is explicitly defined as \( \log(z) = \log|z| + i\theta \), with \(-\pi < \theta < \pi \). Be sure to explain why the strict inequalities are required for continuity.)

2. Determine the value(s) of \( \log((1 + i)^{2i}) \).

3. (a) Set \( w = \cos(z) \) and \( \zeta = e^{iz} \). Show that \( \zeta = w + \sqrt{w^2 - 1} \), where \( \sqrt{w^2 - 1} = \{ \xi \mid \xi^2 = w^2 - 1 \} \).

(b) Show that \( \cos^{-1}(w) = -i \log(w + \sqrt{w^2 - 1}) \), where both sides of the identity are to be interpreted as subsets of \( \mathbb{C} \).

4. Let \( h : [0, 1] \to \mathbb{C} \) be a continuous complex-valued function \( h(t) \) on the unit interval. Define \( H(z) := \int_{0}^{1} \frac{h(t)}{t - z} \, dt, \quad z \in \mathbb{C} \setminus [0, 1] \)

Show that \( H(z) \) is analytic on \( \mathbb{C} \setminus [0, 1] \) and compute its derivative. (Hint: Differentiate ‘by hand’; that is, by using the limit definition of the complex derivative.)

5. Given an analytic function \( f(z) = u(z) + iv(z) \) (with \( u, v \) real-valued), we can write \( z \) in polar form \( z = re^{i\theta} \) and define \( u(r, \theta) := u(re^{i\theta}) \) and \( v(r, \theta) := v(re^{i\theta}) \).

(a) Derive the polar form of the Cauchy-Riemann equations for \( u \) and \( v \):

\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}.
\]

(If you are studying a physical science, you should check that this definition yields consistent units across both equations.)

(b) Check that for any integer \( m \), the functions \( u(r, \theta) = r^m \cos(m\theta) \) and \( v(r, \theta) = r^m \sin(m\theta) \) satisfy (the polar form of) the Cauchy-Riemann equations.

6. Recall that \( \cos^{-1}(z) = -i \log \left[ z + \sqrt{z^2 - 1} \right] \) (with \( \sqrt{z^2 - 1} = \{ w \mid w^2 = z^2 - 1 \} \)). Suppose \( g(z) \) is an analytic branch of \( \cos^{-1}(z) \), defined on a domain \( D \).

(a) Determine \( g'(z) \).

(b) Do different branches of \( \cos^{-1}(z) \) have the same derivative?

7. Let \( f(z) \) be a bounded analytic function, defined on a bounded domain \( D \) in the complex plane, and suppose that \( f(z) \) is one-to-one. Show that the area of \( f(D) \) is given by

\[
\text{Area}(f(D)) = \int_{D} |f'(z)|^2 \, dx \, dy
\]

(Hint: Review the change of variables theorem from your vector calculus course.)
8. Show that Laplace’s equation in polar coordinates is

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

(If you are studying a physical science, you should check that this definition yields consistent units for each term in the equation.)

9. (a) Using Laplace’s equation in polar coordinates, show that \( u(r, \theta) = \theta \log(r) \) is harmonic. (Remember that \( r > 0 \).)

(b) Using polar form of the Cauchy-Riemann equations, to find a harmonic conjugate \( v(r, \theta) \) for \( u(r, \theta) \). What is the corresponding analytic function \( f(z) = u(z) + iv(z) \)?

10. Given a positive number \( B \) with \( 0 < B < \pi \). Find a conformal map of the wedge \( \{ z \mid -B < \arg(z) < B \} \) onto the right half-plane \( \{ w \mid \Re(w) > 0 \} \).

11. The inversion mapping \( f(z) = \frac{1}{z} \) is defined on the extended complex plane \( \mathbb{C}^* = \mathbb{C} \cup \{ \infty \} \) by defining \( f(0) = \infty \), \( f(\infty) = 0 \), and \( f(1) = 1 \). (Remember that this does not mean that \( \infty \) is a complex number.)

(a) Show that the image of a straight line under \( f \) is a circle or a straight line, depending on whether or not the line passes through the origin.

(b) Show that the image of a circle under \( f \) is a straight line or a circle, depending on whether or not the line passes through the origin.

Notice that both circles and straight lines in the complex plane \( \mathbb{C} \) correspond to circles on the Riemann sphere \( S \).

12. Let \( P(x, y) \) and \( Q(x, y) \) be continuous complex-valued functions on a curve \( \gamma \), and define a function \( F(w) \) on \( \mathbb{C} \setminus \gamma \) by

\[
F(w) := \int_{\gamma} \frac{P(x, y)}{z - w} \, dx + \int_{\gamma} \frac{Q(x, y)}{z - w} \, dy, \quad \text{where} \quad z = x + iy.
\]

(a) Show that \( F(w) \) is analytic for \( w \in \mathbb{C} \setminus \gamma \).

(b) Express \( F'(w) \) as a line integral over \( \gamma \).