Math 142A Homework Assignment 1
Due 11:00pm Thursday, January 18, 2024

1. (a) Show $|b| \le a$ if and only if $-a \le b \le a$.
   (b) Prove $\big|\,|a| - |b|\,\big| \le |a - b|$ for all $a, b \in \mathbb{R}$.

2. (a) Prove that $|a + b + c| \le |a| + |b| + |c|$.
   (b) Use induction to prove that $|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$ for any n numbers $a_1,\ a_2,\ \ldots\ ,\ a_n$.

3. Let $a, b \in \mathbb{R}$. Show that if $a < b_1$ for every $b_1 > b$, then $a \le b$.

4. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n} < a < n$.

5. Let S and T be bounded subsets of $\mathbb{R}$.
   (a) Prove that if $S \subseteq T$, then $\inf(T) \le \inf(S) \le \sup(S) \le \sup(T)$.
   (b) Prove that $\sup(S \cup T) = \max\{\sup(S),\ \sup(T)\}$. Note: Be sure that you don't assume that $S \subseteq T$.

6. Let $\mathbb{I}$ be set of real numbers that are not rational; that is, $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. The elements of $\mathbb{I}$ are called *irrational numbers*. Prove that if $a < b$, then there exists $x \in \mathbb{I}$ with $a < x < b$.

7. Prove that the following statements are equivalent:
   (a) $|a - b| < c$,
   (b) $b - c < a < b + c$,
   (c) $a \in (b - c,\ b + c)$.

8. Let $a, b \in \mathbb{R}$. Show that if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \le b$.

9. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ with $a \in A$ and $b \in B$; that is, $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.
   (a) Prove that $\sup(A + B) = \sup(A) + \sup(B)$.
   (b) Prove that $\inf(A + B) = \inf(A) + \inf(B)$.

10. Exhibit an example of:
    (a) a sequence (x_n) of irrational numbers having a limit which is a rational number; that is, $(x_n) \subset \mathbb{I}$ with $\lim\limits_{n\to\infty} x_n = r \in \mathbb{Q}$.
    (b) a sequence (r_n) of rational numbers having a limit which is a irrational number; that is, $(r_n) \subset \mathbb{Q}$ with $\lim\limits_{n\to\infty} r_n = x \in \mathbb{I}$.