1. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous at \( x_0 \) with \( f(x_0) > 0 \). Prove that there is a natural number \( n \) for which \( f(x) > 0 \) for all \( x \) in the interval \( I := (x_0 - 1/n, x_0 + 1/n) \).

2. A function \( f : D \to \mathbb{R} \) is said to be a Lipschitz function if there is a \( C \geq 0 \) such that \(|f(u) - f(v)| \leq C |u - v|\) for all \( u, v \) in \( D \). Prove that a Lipschitz function is continuous.

3. Let \( f : \mathbb{R} \to \mathbb{R} \) have the property that \( f(u + v) = f(u) + f(v) \) for all \( u \) and \( v \).
   
   (a) Let \( m := f(1) \). Prove that \( f(x) = mx \) for all rational numbers \( x \).
   
   (b) Prove that if \( f \) is continuous, then \( f(x) = mx \) for all \( x \).

4. Let \( S \) be a nonempty set of real numbers that is not sequentially compact. Prove that there is an unbounded sequence in \( S \) or there is a sequence in \( S \) that converges to a point \( x_0 \) which is not in \( S \).

5. Let \( f : [0,1] \to \mathbb{R} \) be continuous with \( f(0) > 0 \) and \( f(1) = 0 \). Prove that there is an \( x_0 \) in \( (0,1) \) such that \( f(x_0) = 0 \) and \( f(x) > 0 \) for all \( x \) in \([0,x_0)\); that is, there is a smallest point in the interval \([0,1]\) at which \( f \) attains the value 0.

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function whose image \( f(\mathbb{R}) \) is bounded. Prove that there is a solution to the equation \( f(x) = x \).

7. Let \( f : [a,b] \to \mathbb{R} \) be continuous. Given a natural number \( k \), let \( x_1, \ldots, x_k \) be points in \([a,b]\). Prove that there is a point \( z \) in \([a,b]\) at which

   \[
   f(z) = \frac{f(x_1) + \cdots + f(x_k)}{k}.
   \]

   [Note: As \( k \to \infty \), this becomes the mean value theorem for integrals (Theorem 6.26).]

8. Given \( f : [0,1] \to \mathbb{R} \) continuous such that \( f([0,1]) \subseteq \mathbb{Q} \). Show that \( f \) is a constant function.

9. Let \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \) be uniformly continuous functions. Define the product function \( fg : D \to \mathbb{R} \) by \((fg)(x) := f(x)g(x)\).

   (a) Show that \( fg \) need not be uniformly continuous.
   
   (b) Prove that if \( f \) and \( g \) are also bounded, then \( fg \) is uniformly continuous.

   **Hint:** Write \( f(u)g(u) - f(v)g(v) = f(u)[g(u) - g(v)] + g(v)[f(u) - f(v)] \).

10. A function \( f : D \to \mathbb{R} \) is called a Lipschitz function if there is a \( C \geq 0 \) such that \(|f(u) - f(v)| \leq C |u - v|\) for all \( u, v \in D \). Prove that if \( f \) is a Lipschitz function, then \( f \) is uniformly continuous.