Math 142A Homework Assignment 7
Due Wednesday, December 6

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Assuming knowledge of the sine and cosine functions from your calculus class, f is differentiable for all $x \in \mathbb{R} \setminus \{0\}$.

   (a) Show that f is differentiable at $x = 0$. (Thus, f is differentiable for all x.)

   (b) Show that $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$.

2. Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly increasing differentiable function with $h'(x) > 0$ for all x and $h(\mathbb{R}) = \mathbb{R}$. Given $f : \mathbb{R} \to \mathbb{R}$ differentiable, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := f\left(h^{-1}(x)\right)$ for all x. Find $g'(x)$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there exists a strictly increasing bounded sequence $\{x_n\}$ such that $f(x_n) \leq f(x_{n+1})$ for all indices n. Show that there is an x_0 at which $f'(x_0) \geq 0$. (Note: f itself need not be monotonically increasing; take, for example, $f(x) = \sin(x)$ and $x_n = \frac{n\pi - 2}{2n}$.)

4. A function $f : \mathbb{R} \to \mathbb{R}$ is called $\begin{cases} \textit{even} & \text{if } f(x) = f(-x) \text{ for all } x, \\ \textit{odd} & \text{if } f(x) = -f(-x) \text{ for all } x. \end{cases}$

   (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and odd, then $f' : \mathbb{R} \to \mathbb{R}$ is even.

   (b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and even, then $f' : \mathbb{R} \to \mathbb{R}$ is odd.

5. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x - x^2 & \text{if } x \in \mathbb{Q}, \\ x + x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   (a) Show that $f'(0) = 1$.

   (b) Show that there is no neighborhood I of the point 0 on which f is monotonically increasing.

6. Let $n \in \mathbb{N}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that the equation $f'(x) = 0$ has at most $n - 1$ solutions $x_j \in \mathbb{R}$, $1 \leq j \leq n - 1$. Prove that the equation $f(x) = 0$ has at most n solutions $x_k \in \mathbb{R}$, $1 \leq k \leq n$.

7. Suppose $f : (-1, 1) \to \mathbb{R}$ has n derivatives and that its n^{th} derivative $f^{(n)} : (-1, 1) \to \mathbb{R}$ is bounded. Suppose further that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Show that there is an $M > 0$ such that $|f(x)| \leq M |x|^n$ for all $x \in (-1, 1)$.

8. Suppose $f : (-1, 1) \to \mathbb{R}$ has n derivatives and that there is an $M > 0$ such that $|f(x)| \leq M |x|^n$ for all $x \in (-1, 1)$. Show that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$.