Math 142B
August 21, 2018
**Question 1** The Mean Value Theorem asserts that if \( f : [a, b] \to \mathbb{R} \) is continuous and its restriction \( f : (a, b) \to \mathbb{R} \) is differentiable, then there is a point \( x_0 \in (a, b) \) at which \( f'(x_0) = \frac{f(b) - f(a)}{b - a} \).

The Mean Value Theorem can be proved by

* A. applying Rolle’s Theorem to 
  \( g(x) = f(x) - f(a) - m(x - a) \) with \( m := \frac{f(b) - f(a)}{b - a} \).

B. applying the Extreme Value Theorem to find a point \( x_0 \in (a, b) \) at which \( f'(x_0) \) has an extreme value.

C. applying the Mean Value Theorem for integrals to find \( x_0 \) so that 
  \( f(x_0) = \frac{1}{b - a} \int_a^b f \).

D. All of the above; A, B, and C are all part of the proof of the Mean Value Theorem.

E. None of the above; the Mean Value Theorem only applies to differentiable functions \( f : [a, b] \to \mathbb{R} \).
Question 2  The Cauchy Mean Value Theorem asserts that if $f, g : [a, b] \to \mathbb{R}$ are continuous and their restrictions $f, g : (a, b) \to \mathbb{R}$ are differentiable with $g'(x) \neq 0$ for all $x \in (a, b)$, then there is a point $x_0 \in (a, b)$ at which $f'(x_0) = \frac{f(b)-f(a)}{g(b)-g(a)}$.

The Cauchy Mean Value Theorem can be proved by

*A. applying Rolle’s Theorem to $h(x) = f(x) - mg(x)$ with the constant $m := \frac{f(b)-f(a)}{g(b)-g(a)}$.

B. applying the Mean Value Theorem to $h(x) = \frac{f(x)-f(a)}{g(x)-g(a)}$.

C. applying the Extreme Value Theorem to $h(x) = \frac{f(x)}{g(x)}$ in order to find a point $x_0$ at which $h'(x_0) = 0$.

D. all of the above; the Cauchy Mean Value Theorem can be proved in many ways.

E. none of the above; Cauchy had nothing to do with the Mean Value Theorem.
Question 3  Given a neighborhood $I$ of $x_0$, $n$ a nonnegative integer, and a function $f : I \to \mathbb{R}$ with $n + 1$ derivatives such that $f^{(k)}(x_0) = 0$ for $0 \leq k \leq n$. Then for each $x \neq x_0$ in $I$, there is a $c$ strictly between $x$ and $x_0$ at which $f(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$.

This statement

A. can be proven by applying the Cauchy Mean Value Theorem inductively to $f^{(k)}(x)$ and $g^{(k)}(x)$ for $0 \leq k \leq n$, starting with $f(x)$ and $g(x) = (x - x_0)^n$.

B. is a special case of the Lagrange Remainder Theorem since $p_n(x) = 0$ is the $n^{th}$ Taylor polynomial for $f$ at $x_0$.

C. immediately implies the Lagrange Remainder Theorem since $R_n(x) = f(x) - p_n(x)$ has contact of order $n$ with the constant function 0.

D. A and B.

*E. A, B, and C.
Question 4  Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function $f(x) = e^x$. Then,

A. The $n^{th}$ Taylor polynomial for $f$ at $x = 0$ is
\[ p_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + \frac{1}{2} + \cdots + \frac{1}{n!} x^n. \]

B. $f^{(k)}(0) = p_n^{(k)}(0)$ for $0 \leq k \leq n$.

C. $\lim_{n \to \infty} p_n(x) = f(x)$ for every $x$.

D. A and B.

*E. A, B, and C.
Question 5  Let \( H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \) be the \( n \)th harmonic number. Then,

A. \( \lim_{n \to \infty} H_n = \infty. \)

B. \( \{c_n\}, \) where \( c_n = H_n - \log(n + 1), \) is monotonically increasing and bounded above by 1.

C. \( \lim_{n \to \infty} c_n = \gamma, \) where \( c_n = H_n - \log(n + 1) \) and \( \gamma \leq 1 \) is called Euler’s constant.

D. B and C.

*E. A, B, and C.
Question 6  Given $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ with derivatives of all orders. Then $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^m$ is the $n^{th}$ Taylor polynomial for $f$ at $x_0$, and

A. $p_n$ has contact of order $n$ with $f$ at $x_0$.  

B. $f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$ for some $c$ strictly between $x$ and $x_0$.  

C. $\lim_{n \to \infty} p_n(x) = f(x)$ for every $x$.  

* D. A and B.  

E. A, B, and C.