Math 142B
August 23, 2018
Question 1  We can write \( f(x) = \log(1 + x) \) in the form

\[
f(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k + R_n(x).
\]

We can say that

A. \( \lim_{n \to \infty} R_n(x) = 0. \)

B. \( R_n(x) = \frac{(-1)^n}{k(1 + c)^{n+1}} x^{n+1} \) for some \( c \) strictly between 0 and \( x. \)

C. \( R_n(x) = \int_1^{1+x} \frac{(1 - t)^n}{t} \, dt. \)

D. All of the above.

*E. All of the above, provided \(-1 < x \leq 1\); otherwise, A is false.
Question 2  Let \( I \) be a neighborhood of \( x_0 \) and let \( f: I \rightarrow \mathbb{R} \) be a function with \( n+1 \) derivatives. Then,

\[
f(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + R_n(x),
\]

where the \( n^{th} \) remainder \( R_n(x) \) is equal to

A. \( \frac{f^{(n+1)}(a)}{(n+1)!} h^{n+1} \) for some \( a \) strictly between \( x_0 \) and \( x_0 + h \).

B. \( \frac{f^{(n+1)}(x_0 + b)}{(n+1)!} h^{n+1} \) for some \( b \) strictly between 0 and \( h \).

C. \( \frac{f^{(n+1)}(c)}{(n+1)!} h^{n+1} \) for some \( c \) strictly between \( x \) and \( x_0 \).

*D. A or B.

E. A, B, or C.

[Note: E would be true if \( x \) were defined to be \( x := x_0 + h \).]
Question 3  A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is integrable if and only if

A.  $\int_{a}^{b} f = \int_{a}^{b} f$.

B.  there is a sequence of partitions $\{P_n\}$ of $[a, b]$ with $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$.

C.  for every $\varepsilon > 0$ there is a corresponding partition $P$ of $[a, b]$ for which $[U(f, P) - L(f, P)] < \varepsilon$

*D.  All of the above; they are equivalent.

E.  None of the above; not all bounded functions are integrable.
Question 4  Given $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ with derivatives of all orders. Then $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ is the $n^{th}$ Taylor polynomial for $f$ at $x = x_0$, and

A. $p_n$ has contact of order $n$ with $f$ at $x_0$.

B. $f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ for some $c$ strictly between $x$ and $x_0$.

C. $\lim_{n \to \infty} p_n(x) = f(x)$ for every $x$.

*D. A and B.*

E. A, B, and C.
Question 5  Given $f : [a, b] \rightarrow \mathbb{R}$ bounded and $\{P_n\}$ a sequence of partitions of $[a, b]$. Then,

A. $U(f, P_n) - L(f, P_n) \geq \int_a^b f - \int_a^b f \geq 0$ for every index $n$.

B. $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$ implies $f$ is integrable.

C. $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$ implies $\lim_{n \to \infty} \text{gap}(P_n) = 0$.

*D. A and B.

E. A, B, and C.
Question 6  Given $f : [a, b] \rightarrow \mathbb{R}$ bounded and $P \subseteq P^*$ partitions of $[a, b]$.

A. $P^*$ is a refinement of $P$.
B. $L(f, P^*) = L(f, P)$ and $U(f, P^*) = U(f, P)$.
C. $L(f, P^*) \geq L(f, P)$ and $U(f, P^*) \leq U(f, P)$.
D. A and B.

*E. A and C.