Question 1  Given a continuous function \( f : [0, 1] \rightarrow \mathbb{R} \). Given \( \varepsilon > 0 \), there is a polynomial \( p : \mathbb{R} \rightarrow \mathbb{R} \) such that \( |f(x) - p(x)| < \varepsilon \) for all \( x \in [0, 1] \). We can construct such a polynomial \( p \) by defining

\[
p(x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k} \quad \text{for all } x \text{ and an appropriately chosen index } n.
\]

A. \( p(x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k} \) for all \( x \) and an appropriately chosen index \( n \).

B. \( p(x) := \sum_{k=0}^{n} \frac{f^{(k)} \left( \frac{1}{2} \right)}{k!} \left( x - \frac{1}{2} \right)^k \) for all \( x \) and an appropriately chosen index \( n \).

C. \( p(x) := \sum_{k=0}^{n} \frac{f^{(k)} (0)}{k!} x^k \) for all \( x \) and an appropriately chosen index \( n \).

D. A and B; both methods produce a polynomial \( p \) with the required property.

E. A, B, and C; all three methods produce a polynomial \( p \) with the required property.
Question 2  Given a continuous function $f : [a, b] \to \mathbb{R}$. Define $g : [0, 1] \to [a, b]$ by $g(t) = a + (b - a)t$. Then,

A. $f \circ g : [0, 1] \to \mathbb{R}$ given by $(f \circ g)(t) = f(g(t))$ is continuous.

B. Given $\varepsilon > 0$, there is a polynomial $q$ such that $|(f \circ g)(t) - q(t)| < \varepsilon$ for all $t \in [0, 1]$.

C. Given $\varepsilon > 0$ and $q$ as in B, set $p(x) = q \left( \frac{x-a}{b-a} \right)$. Then, $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$

* D. All of the above.

E. None of the above.
Question 3  Given any number $\beta$. Define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = (1 + x)^\beta$. Then,

* A. $f^{(k)}(x) = \beta(\beta - 1) \cdots (\beta - k + 1)(1 + x)^{\beta-k}$.

B. $f(x) = \sum_{k=0}^{\infty} \frac{\beta(\beta - 1) \cdots (\beta - k + 1)}{k!} (1 + x)^{\beta-k}$.

C. $f(x) = \sum_{k=0}^{\infty} \binom{\beta}{k} (1 + x)^{\beta-k}$.

D. B and C; they are the same since $\binom{\beta}{k} = \frac{\beta(\beta-1) \cdots (\beta-k+1)}{k!}$.

E. All of the above.
Question 4  Given a neighborhood $I$ of $x_0$ and an infinitely differentiable function $f : I \to \mathbb{R}$. Then,

A. $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(x_0)(x - t)^n \, dt,$
   for every $x \in I$ and every index $n$.

B. For every $x \in I$ and every index $n$, there is a $c$ strictly between $x$ and $x_0$ at which
   $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$.

C. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ for every $x \in I$ and every index $n$.

D. All of the above.

*E. A and B.
Question 5  The function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \end{cases}$$

is an example of a function that is

A. bounded.
B. infinitely differentiable.
C. analytic.
D. All of the above.

*E. A and B.
Question 6 Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function $f(x) = e^x$. Then,

A. The $n^{th}$ Taylor polynomial for $f$ at $x = 0$ is

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + \frac{1}{2} + \cdots + \frac{1}{n!} x^n.$$ 

B. $f^{(k)}(0) = p^{(k)}_n(0)$ for $0 \leq k \leq n$.

C. $\lim_{n \to \infty} p_n(x) = f(x)$ for every $x$.

D. A and B.

*E. A, B, and C.