Math 142B
August 29, 2018
Question 1  Given a neighborhood $I$ of a point $x_0$ and an infinitely differentiable function $f : I \to \mathbb{R}$. Then, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$ for every $x \in I$ whenever

A. $\lim_{n \to \infty} [f(x) - p_n(x)] \to 0$ for every $x \in I$, where $p_n$ is the $n^{th}$ Taylor polynomial for $f$ at $x_0$.

B. There is an $M > 0$ for which $\left| f^{(k)}(x) \right| \leq M^k$ for every $x \in I$ and every index $k$.

C. $\lim_{n \to \infty} \frac{f^{(n+1)}(x)}{(n+1)!} (x - x_0)^{n+1} = 0$ for every $x \in I$.

*D. A and B.*

E. A, B and C.

[Note: Whether or not C is true is a Math 142B open question.]
Question 2  Given any number $\beta$. Define $f : (-1, 1) \to \mathbb{R}$ by
$f(x) = (1 + x)^\beta$. Then,

A. $f^{(k)}(x) = \beta(\beta - 1) \cdots (\beta - k + 1)(1 + x)^{\beta-k}$, thus
$f^{(k)}(0) = \beta(\beta - 1) \cdots (\beta - k + 1)$.

B. $f(x) = \sum_{k=0}^{\infty} \frac{\beta(\beta - 1) \cdots (\beta - k + 1)}{k!} x^k$.

C. $f(x) = \sum_{k=0}^{\infty} \binom{\beta}{k} x^k$.

D. B and C; they are the same since $\binom{\beta}{k} = \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$.

E. All of the above.
Question 3  For each index $n$, let $f_n : [0, 1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^n} \text{ for some integer } k, \ 0 \leq k \leq 2^n \\ 0 & \text{otherwise} \end{cases}$$

Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in [0, 1]$. Then,

A. $\int_0^1 f_n \, dx = 0$ for every index $n$.

B. $\int_0^1 f \, dx = 0$ and $\int_0^1 f = 1$.

C. $\int_0^1 f = 0$.

*D. A and B.*

E. A and C.
Question 4  For each index $n$, let $f_n : [0, 1] \rightarrow \mathbb{R}$ be given by

$$ f_n(x) = \begin{cases} 
  n^2x & \text{if } 0 \leq x < \frac{1}{n} \\
  2n - n^2x & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\
  0 & \text{if } \frac{2}{n} \leq x \leq 1 
\end{cases} $$

Let $f : [0, 1] \rightarrow \mathbb{R}$ be given by $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in [0, 1]$. Then,

A. $\int_{0}^{1} f_n = 1$ for every index $n$.

B. $\int_{0}^{1} f = 0$.

C. $\int_{0}^{1} f = \lim_{n \to \infty} \int_{0}^{1} f_n$.

* D. A and B.

E. A and C.
Question 5  Given a sequence of functions \( \{f_n : [a, b] \rightarrow \mathbb{R} \} \) such that \( \{f_n\} \) converges pointwise to \( f \) on \( [a, b] \). Then we can say that

A. if \( f_n \) is integrable for every index \( n \), then \( f \) is integrable.

B. if \( \int_a^b f_n = 1 \) for every index \( n \), then \( \int_a^b f = 1 \).

C. if \( f_n \) is continuous for every index \( n \), then \( f \) is continuous.

D. All of the above.

*E. None of the above.
Question 6  Given a neighborhood $I$ of a point $x_0$ and a function $f : I \to \mathbb{R}$ with $n + 1$ derivatives. Let $p_n(x)$ be the $n^{th}$ Taylor polynomial for $f$ at $x = x_0$. Then,

A. There is a $c$ strictly between $x$ and $x_0$ at which

$$f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!} (x - x_0)^{n+1}.$$

B. There is a $c$ strictly between $x$ and $x_0$ at which

$$\frac{f^{(n+1)}(c)}{(n + 1)!} (x - x_0)^{n+1} = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n \, dt.$$

C. $\lim_{n \to \infty} \left\{ \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n \, dt \right\} = 0$.

*D. A and B.*

E. A, B, and C.