1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function with a second derivative and that satisfies

\[
\begin{align*}
  f''(x) + f(x) &= e^{-x} \quad \text{for all } x, \\
  f(0) &= 0 \quad \text{and} \quad f'(0) = 2.
\end{align*}
\]

Find the fourth Taylor polynomial for \( f : \mathbb{R} \to \mathbb{R} \) at \( x = 0 \).

2. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) has three derivatives and that the third Taylor polynomial for \( f \) at \( x = 0 \) is \( p_3(x) = 1 + 4x - x^2 + \frac{1}{6}x^3 \).

Show that there is a neighborhood \( I \) of 0 such that \( f : I \to \mathbb{R} \) is (1) positive, (2) strictly increasing, and (3) has a strictly decreasing derivative.

3. Prove that \( 1 + \frac{1}{2}x - \frac{1}{8}x^2 < \sqrt{1 + x} < 1 + \frac{1}{2}x \) for all \( x > 0 \).

4. Prove that \( 1 + \frac{1}{3}x - \frac{1}{9}x^2 < (1 + x)^{\frac{1}{3}} < 1 + \frac{1}{3}x \) for all \( x > 0 \).

5. Write the polynomial \( p(x) = x^5 - x^3 + x \) in the form \( p(x) = \sum_{k=0}^{5} a_k (x-1)^k \).

6. Prove that \( |\sin(x+h) - \sin(x) - h \cos(x)| \leq \frac{1}{2}h^2 \) for every pair of numbers \( x \) and \( h \).

7. Use the fact that the \( n \)th Taylor polynomial at \( x = x_0 \) for a polynomial \( p \) of degree at most \( n \) is itself [that is, \( p_n(x) = p(x) \)] to show that if the point \( x_0 \) is a zero of a polynomial \( p \) [that is, \( p(x_0) = 0 \)], then there is a polynomial \( q \) such that \( p(x) = (x-x_0)q(x) \) for all \( x \).

8. A number \( x_0 \) is said to be a zero of order \( k \) of a polynomial \( p \) provided that \( k \) is a natural number such that \( p(x) = (x-x_0)^k r(x) \), where \( r(x) \) is a polynomial with \( r(x_0) \neq 0 \). Prove that \( x_0 \) is a zero of order \( k \) of a polynomial \( p \) if and only if

\[
p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0 \quad \text{and} \quad p^{(k)}(x_0) \neq 0.
\]

9. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) both be functions with \( n + 1 \) continuous derivatives. Prove that \( f \) and \( g \) have contact of order \( n \) at \( x = 0 \) if and only if \( \lim_{x \to 0} \frac{f(x) - g(x)}{x^n} = 0 \).

10. Let \( I \) be a neighborhood of the point \( x_0 \) and let \( f : I \to \mathbb{R} \) be a function with a positive continuous third derivative \( f'''(x) > 0 \) for all \( x \in I \).

(a) Prove that if \( x_0 + h \neq x_0 \) is in \( I \), there is a unique number \( \theta = \theta(h) \in (0, 1) \) for which

\[
f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(x_0 + \theta h)\frac{h^3}{6}.
\]

(b) Prove that \( \lim_{h \to 0} \theta(h) = \frac{1}{3} \).