1. (a) Prove that the dyadic rationals are dense in $\mathbb{R}$.  
[Hint: Adapt the proof of Theorem 1.9 in your text.]
(b) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by
$$f_n(x) = \begin{cases} 
1 & \text{if } x = \frac{k}{2^n} \text{ for some integer } k, 0 \leq k \leq 2^n \\
0 & \text{otherwise}
\end{cases}$$
Show that $f_n$ is integrable for every index $n$.
(c) $\{f_n\}$ converges pointwise on $[0, 1]$ to the function $f$ defined by
$$f(x) = \begin{cases} 
1 & \text{if } x \text{ is a dyadic rational} \\
0 & \text{otherwise}
\end{cases}$$
Use the result of part (a) to show that $f$ is not integrable.

2. For each nonnegative integer $n$ define $p_n(x) : (-1, 1) \to \mathbb{R}$ by $p_n(x) = \sum_{k=0}^{n} x \left( 1 - x^2 \right)^k$. Show that $\{p_n\}$ converges pointwise on $(-1, 1)$ to a function $p : (-1, 1) \to \mathbb{R}$.

3. For each nonnegative integer $n$, define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = \frac{1}{nx + 1}$.
(a) Find the function $f : [0, 1] \to \mathbb{R}$ to which $\{f_n\}$ converges pointwise.
(b) Show that the convergence of $\{f_n\}$ to $f$ is not uniform on $[0, 1]$.

4. For each nonnegative integer $n$, define $g_n : [0, 1] \to \mathbb{R}$ by $g_n(x) = \frac{x}{nx + 1}$.
(a) Find the function $g : [0, 1] \to \mathbb{R}$ to which $\{g_n\}$ converges pointwise.
(b) Show that the convergence of $\{g_n\}$ to $g$ is uniform on $[0, 1]$.

5. For each index $n$, suppose that the function $f_n : \mathbb{R} \to \mathbb{R}$ is bounded and that the sequence $\{f_n\}$ converges uniformly to a function $f$ on $\mathbb{R}$. Prove that the limit function $f : \mathbb{R} \to \mathbb{R}$ is also bounded.

6. Given a bounded sequence $\{a_k\}$ of real numbers, for each index $n$ and real number $x$ define $f_n(x) = \sum_{k=0}^{n} \frac{a_k}{k!} x^k$. Prove that for each $r > 0$, the sequence of functions $\{f_n : [-r, r] \to \mathbb{R}\}$ is uniformly convergent.

7. For each index $n$, define $f_n : (-1, 1) \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ and define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = |x|$.
(a) Prove that $\{f_n\}$ converges uniformly to $f$ on $(-1, 1)$.
(b) Verify that each $f_n$ is continuously differentiable, but $f$ is not differentiable at $x = 0$.
(c) Explain why this does not contradict the textbook’s Theorem 9.33.
8. For each index \( n \), define \( g_n : [0, 1] \to \mathbb{R} \) by \( g_n(x) = nxe^{-nx^2} \).

(a) Prove that \( \{g_n\} \) converges pointwise on \( [0, 1] \) to the constant function 0.

(b) Show that the sequence of integrals \( \{\int_0^1 g_n\} \) does not converge to 0.

(c) Explain why this does not contradict the textbook’s Theorem 9.32.

9. Given \( g : [a, b] \to \mathbb{R} \) and \( h : [a, b] \to \mathbb{R} \) bounded functions with \( g(x) \leq h(x) \) for all \( x \in [a, b] \).

(a) Show that if \( P \) is a partition of \( [a, b] \), then \( L(g, P) \leq L(h, P) \) and \( U(g, P) \leq U(h, P) \).

(b) Show that \( \int_a^b g \leq \int_a^b h \) and \( \int_a^b g \leq \int_a^b h \).

10. Prove that \( x = \sum_{k=0}^{\infty} \left(1 - \frac{1}{x}\right)^k \) if \( |1 - x| < |x| \).

11. Define \( f : (-1, 1) \to \mathbb{R} \) by \( f(x) = \frac{1}{(1 - x)^3} \). Find a power series expansion for \( f \) valid on \((-1, 1)\).

12. Given a sequence \( \{a_n\} \) such that \( \lim_{n \to \infty} |a_n|^{\frac{1}{n}} = \alpha \).

(a) If \( \alpha > 0 \), show that \( \sum_{n=0}^{\infty} a_n x^n \) converges if \( |x| < \frac{1}{\alpha} \) and diverges if \( |x| > \frac{1}{\alpha} \).

(b) If \( \alpha = 0 \), show that \( \sum_{n=0}^{\infty} a_n x^n \) converges for all \( x \).