Instructions

1. Write your Name and PID in the spaces provided above.
2. Make sure your Name is on every page.
3. No calculators, tablets, phones, or other electronic devices are allowed during this exam.
4. You may use two handwritten pages of notes, but no books or other assistance during this exam.
5. Read each question carefully and answer each question completely.
6. Write your solutions clearly in the spaces provided.
7. Show all of your work. No credit will be given for unsupported answers, even if correct.

0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.
1. (6 points) (a) Let \( A = \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 3 & 1 & 1 & -1 & h \\ 0 & 1 & -5 & 2 & -5 \end{bmatrix} \). Determine all values of \( h \) for which \( A \) is not full rank. What is the dimension of the null space of \( A \) for those values of \( h \)?

(b) Find the solution (in parametric form) of the following system of equations:

\[
\begin{align*}
    x_1 + 2x_3 &= -1 \\
    3x_1 + x_2 + x_3 &= -1 \\
    x_2 - 5x_3 &= 2
\end{align*}
\]
2. The matrix \( A = \begin{bmatrix} 2 & 3 & 1 & 8 & 2 & 3 \\ -1 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & -2 & 5 & 3 & 8 \\ 1 & 4 & 3 & 9 & 2 & 1 \end{bmatrix} \) is row equivalent to \( \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) Find a basis for \( \text{Col}(A) \).

(b) Find a basis for \( \text{Row}(A) \).

(c) Find a basis for \( \text{Row}(A)^\perp \).
3. In each of the following examples, a vector space $V$ is given, along with a subset $S \subseteq V$. Indicate whether the set $S$ is a subspace of $V$ or not, by filling in the $\bigcirc$ YES bubble if it is a subspace, or the $\bigcirc$ NO bubble if it is not. 2 points will be assigned for each correct response, 1 point for each blank non-response, and 0 points for each incorrect response. No justification is required.

(a) $V = \mathbb{R}^2$, and $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \leq y \right\}$.

$\bigcirc$ YES $\bigcirc$ NO

(b) $V = \mathbb{R}^n$, $W \subseteq V$ is a subspace, and $S = W^\perp$.

$\bigcirc$ YES $\bigcirc$ NO

(c) $V = \mathbb{R}^3$, and $S$ is the set of vectors in $V$ with length 1: $S = \{ v \in V : \|v\| = 1 \}$.

$\bigcirc$ YES $\bigcirc$ NO

(d) $V = M_{4 \times 4}$ is the space of $4 \times 4$ matrices, and $S$ is the set of skew-symmetric matrices: $S = \{ A \in M_{4 \times 4} : A^\top = -A \}$.

$\bigcirc$ YES $\bigcirc$ NO

(e) $V = \mathbb{P}_3$ is the space of polynomials of degree $\leq 3$, and $S = \{ p \in \mathbb{P}_3 : p(1) = 1 \}$.

$\bigcirc$ YES $\bigcirc$ NO
4. Consider the matrix
\[
A = \begin{bmatrix}
1 & 2 & 2 \\
0 & 1 & 0 \\
1 & 0 & 2
\end{bmatrix}
\]

(a) Find all the eigenvalues of $A$. Is $A$ diagonalizable? Explain your answer.

(b) Determine the eigenspace of $A$ for the largest eigenvalue, 3.

(c) What is the largest eigenvalue of the matrix $A^2$? And what is its eigenspace?
5. (7 points) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

(a) Calculate the determinant of $A$, showing your work. Use this to show that $A$ is invertible.

(b) Calculate the inverse of $A$, showing your work.

(c) Explain why there is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$. 
(6 points) 6. Consider the matrix 
\[
A = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
2 & 3 \\
-2 & -1
\end{bmatrix}.
\]

(a) Use the Gram–Schmidt process to find an orthonormal basis \{\hat{u}_1, \hat{u}_2\} for \text{Col}(A).

(b) Use the result in part (a) to factorize \( A = QR \) where \( Q \in M_{4 \times 2} \) has orthonormal columns and \( R \in M_{2 \times 2} \) is upper-triangular.
7. (6 points) Suppose $A$ and $B$ are square matrices, and suppose that $AB = BA$. Let $v$ be an eigenvector of $A$ with eigenvalue $\lambda$.

(a) Show that $u = Bv$ is also an eigenvector of $A$, with eigenvalue $\lambda$.

(b) Let $v$ be an eigenvector of $A$ with eigenvalue $\lambda$, as above. Suppose that the algebraic multiplicity of the eigenvalue $\lambda$ is 1. Use (a) to show that $v$ is also an eigenvector of $B$. 