Math 10C Calculus **Practice Midterm I Solutions**

1. Suppose $f(x)$ is a function whose 5^{th}-degree Taylor polynomial about $x = 0$ is

$$P_5(x) = x - x^2 + x^3 - x^4 + x^5.$$

(a) (5 points) Is f increasing or decreasing at $x = 0$?
$f'(0) = 1$ (the coefficient of x), so f is increasing at $x = 0$.

(b) (5 points) What is the concavity of f at $x = 0$?
$f''(0) = -2$, so f is concave down at $x = 0$.

(c) (5 points) What is the value of $f^{(4)}(0)$?
$f^{(4)}(0) = -24$

(d) (5 points) Suppose you are also given that $f^{(6)}(0) = -72$. Find the 6^{th}-degree Taylor polynomial for $f(x)$ about $x = 0$.
$\frac{-72}{6!} = -\frac{72}{720} = -\frac{1}{10}$, so $P_6(x) = x - x^2 + x^3 - x^4 + x^5 - \frac{1}{10}x^6$.

2. Suppose that a is some fixed value with $|a| < 1$. We define the sums S_n as follows:

$$\begin{aligned} S_1 &= a \\ S_2 &= a - a^2 \\ S_3 &= a - a^2 + a^3 \\ S_4 &= a - a^2 + a^3 - a^4 \\ &\vdots \end{aligned}$$

(a) (8 points) Is S_n a geometric series? Explain your answer by referring to features of a geometric series.
S_n is a geometric series with initial term a and ratio $-a$.

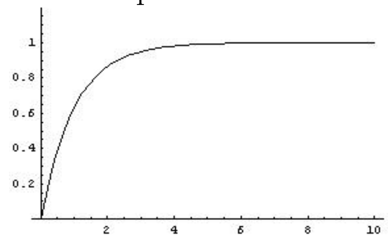
(b) (6 points) Suppose now that $a = 2$. What is the value of S_{10}?
$S_{10} = 2\left(\frac{1-(-2)^{10}}{1-(-2)}\right) = \frac{2}{3}(1 - 2^{10}) = -682$

(c) (6 points) Suppose instead that $a = \frac{1}{5}$. What is the value of the infinite series

$$S = a - a^2 + a^3 - a^4 + a^5 - a^6 + a^7 - \cdots?$$

$S = \frac{1}{5}\left(\frac{1}{1-(-\frac{1}{5})}\right) = \frac{1}{6}$

3. The graph of $f(x) = 1 - e^{-x}$ is given below. You are informed that $f(x)$ is either a probability density function or a cumulative distribution function for some statistical experiment.



(a) (7 points) If you think $f(x)$ is a density function, sketch a graph for the corresponding distribution function. If you think $f(x)$ is a distribution function, sketch the corresponding density function.
The corresponding density function is $\frac{d}{dx}(1 - e^{-x}) = e^{-x}, x \geq 0$. (The domain is restricted to $x \geq 0$ since $f(0) = 0$.)

(b) (7 points) Find the mean value for the outcome of the experiment.

$$\begin{aligned}
\mu &= \int_0^\infty xe^{-x}dx \\
&= \lim_{b\to\infty} \int_0^b xe^{-x}dx \\
&= \lim_{b\to\infty} \left(-xe^{-x} - e^{-x}\right)_0^b \\
&= \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right) \\
&= 1
\end{aligned}$$

(c) (6 points) To what extent is the following statement true? Explain.

"If $p(x)$ is the density function for an experiment and $p(2) = 0.86$, then we can conclude that the outcome $x = 2$ occurs 86% of the time."
This statement is false. What is true is that if we look at an interval of width Δx about $x = 2$, then the percentage of outcomes falling withing that interval is approximately $0.86\Delta x$.

4. Suppose $f(x, y) = 5 - x^2 - y^2$.

(a) (6 points) Describe, in words and with a graph, the cross-sections of the surface $z = f(x, y)$ with x fixed.
If we fix $x = k$, we get cross-sections with equations $z = 5 - k^2 - y^2$. These are parabolas, opening downward.

(b) (6 points) Describe, in words and with a graph, the cross-sections of the surface $z = f(x, y)$ with z fixed.
If we fix $z = k$, we get cross-sections with equations $k = 5 - x^2 - y^2$, i.e., $x^2 + y^2 = 5 - k$. These are circles centered at the origin.

(c) (8 points) Sketch a contour diagram for the function $f(x, y)$ with four labeled contours.
As an example, the contour with $z = 1$ is the circle $x^2 + y^2 = 4$.

5. (a) (5 points) Find the equation of the linear function $z = c + mx + ny$ whose graph contains the points $(0, 0, 0)$, $(0, 2, -1)$ and $(-3, 0, -4)$.
The slope in the x direction is $m = \frac{-4-0}{-3-0} = \frac{4}{3}$. The slope in the y direction is $n = \frac{-1-0}{2-0} = -\frac{1}{2}$. Using the point $(0, 0, 0)$, we have $z = \frac{4}{3}x - \frac{1}{2}y$.

(b) (5 points) Find the equation of the linear function $z = c + mx + ny$ whose graph intersects the xz-plane in the line $z = 3x + 4$ and intersects the yz-place in the line $z = y + 4$.
The slope in the x direction is $m = 3$ and the slope in the y direction is $n = 1$. Using the point $(0, 0, 4)$, we have $z = 4 + 3x + y$.

(c) (5 points) Is it possible to determine the equation for a plane, if we know that the points $(7, 1, 2)$, $(8, 3, 6)$ and $(9, 5, 10)$ lie on the plane? Explain.
No. All three points lie on the same line. (To get from the first point to the second point we move 1 in the (positive) x direction, 2 in the y direction, and 4 in the z direction. The exact same movements get us from the second point to the third point.)

(d) (5 points) Is it possible to determine the equation for a plane, if we know that the points $(5, 1, 7)$ and $(8, 1, 8)$ lie on the plane, and also that the slope in the x-direction is $\frac{1}{3}$? Explain.
No. From the first piece of information we see that the slope in the x direction is $m = \frac{8-7}{8-5} = \frac{1}{3}$. Thus the second piece of information is redundant. Without knowing the slope in the y direction, we cannot determine the equation of the plane.