Name: Solutions
SID:

20C, Ferrara, Oct 19, Version 2

Midterm 1

This exam has 7 pages and 6 problems. Make sure that your exam has all 7 pages and that your name is on every page.
Put your name and student ID on every page.
You must show your work and justify your answers to receive full credit unless otherwise stated.
If you need more space, use the pack of the pages; clearly indicate when you have done this.
You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

1	
2	
3	
4	
5	
6	
7	
Total	

Name: ____________________

20C, Ferrara, Oct 19, Version 2

SID: ____________________

1. Let $\mathbf{v} = (4,1,-1), \mathbf{w} = (-4,2,4), \alpha = 7$. Calculate the following (use the space at the bottom of the page for your work):

   (a) $\mathbf{v} + \mathbf{w} =$ (0,3,3)

   (b) $\alpha\mathbf{v} =$ (28,7,-7)

   (c) $\mathbf{v} \cdot \mathbf{w} =$ -18

   (d) $\|\mathbf{v}\| =$ $\sqrt{18} = 3\sqrt{2}$

   (e) $\|\mathbf{w}$ = 6

   (f) $\mathbf{v} \times \mathbf{w} =$ (6,-12,12)

   (g) $\mathbf{w} \times \mathbf{v} =$ $-\vec{v} \times \vec{w} =$ (-6,12,-12)

   (h) $v \cdot (\mathbf{v} \times \mathbf{w}) =$ 0

   (i) $w \cdot (\mathbf{v} \times \mathbf{w}) =$ 0

   cross product is orthogonal to $\vec{v}$ and $\vec{w}$

   (j) Angle between $\mathbf{v}$ and $\mathbf{w}$ is $3\pi/4$.

TYPOS

$\vec{v} + \vec{w} = (4,1,-1) + (-4,2,4) = (0,3,3)$

$\alpha\vec{v} = 7 \cdot (4,1,-1) = (28,7,-7)$

$\vec{v} \cdot \vec{w} = (4,1,-1) \cdot (-4,2,4) = -16 + 2 - 4 = -18$

$\|\vec{v}\| = \sqrt{4^2 + 1^2 + (-1)^2} = \sqrt{18} = 3\sqrt{2}$, $\|\vec{w}\| = \sqrt{(-4)^2 + 2^2 + 4^2} = \sqrt{16+4+16} = \sqrt{36} = 6$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 4 & 1 & -1 \\ -4 & 2 & 4 \end{vmatrix} = (1\cdot 4 - (-1)\cdot 2)i - (4\cdot 4 - (-1)\cdot(-4))j + (4 \cdot 2 - (1)\cdot(-4))k$$

$$= (4+2)i - (16-4)j + (8+4)k = (6, -12, 12)$$

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$

$-18 = 3\sqrt{2} \cdot 6 \cos\theta$

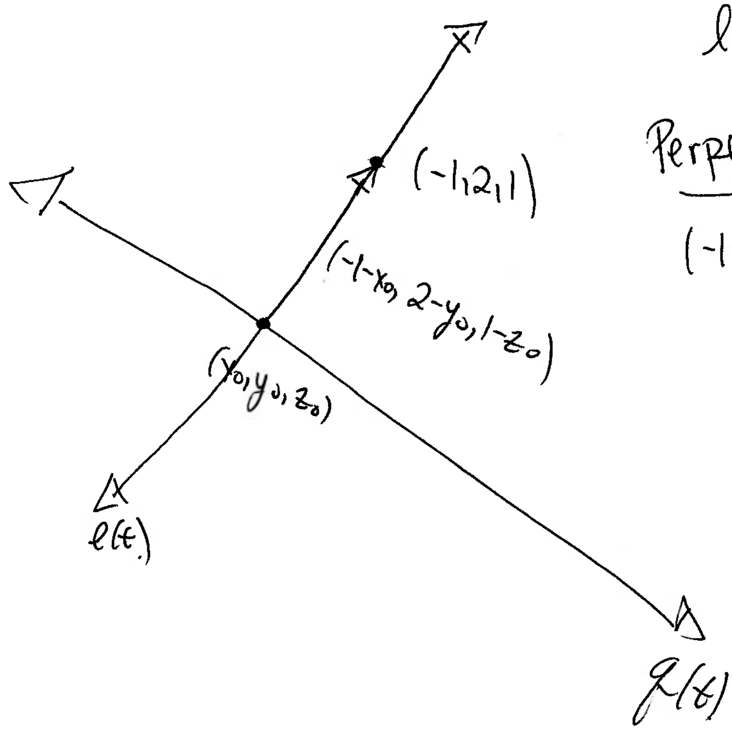
$-\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} = \cos\theta$, $\theta = 3\pi/4$

20C, Ferrara, Oct 19, Version 2

Name: ____________________

SID: ____________________

2. Find the equation of the line that contains the point $(-1, 2, 1)$, and is perpendicular and intersecting the line with equation $q(t) = (3, 4, 3) + t(1, 3, 0)$.



$l(t) = (x_0, y_0, z_0) + t(-1 - x_0, 2 - y_0, 1 - z_0)$

Perpendicular:

$(-1 - x_0, 2 - y_0, 1 - z_0) \cdot (1, 3, 0) = 0$

$-1 - x_0 + 3(2 - y_0) + 0 \cdot (1 - z_0) = 0$

$-1 - x_0 + 6 - 3y_0 = 0$

$x_0 = 5 - 3y_0$

Need to solve for x_0, y_0, z_0

Intersecting: $q(t_0) = (x_0, y_0, z_0)$ for some t_0

$(3, 4, 3) + t_0(1, 3, 0) = (x_0, y_0, z_0)$

$3 + t_0 = x_0$

$4 + 3t_0 = y_0$

$3 = z_0$

plug $x_0 = 5 - 3y_0$ into $3 + t_0 = x_0$

$3 + t_0 = 5 - 3y_0$

$t_0 = 2 - 3y_0$

plug $t_0 = 2 - 3y_0$ into $4 + 3t_0 = y_0$

$4 + 3(2 - 3y_0) = y_0$

$4 + 6 - 9y_0 = y_0$

$10 = 10y_0$

$1 = y_0$

$x_0 = 5 - 3y_0 = 5 - 3 = 2$

Answer:

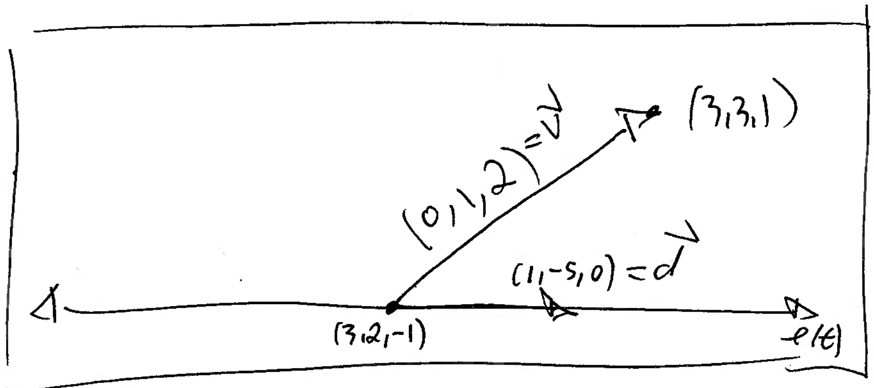
$l(t) = (2, 1, 3) + t(-3, 1, -2)$

20C, Ferrara, Oct 19, Version 2

Name: ____________________

SID: ____________________

3. Find the equation of the plane contianing the line $\ell(t) = (3+t, 2-5t, -1)$ and the point $(3, 3, 1)$.



To get the normal $\vec{n}$ cross the direction vector, $\vec{d}$, of $\ell(t)$ with a vector $\vec{v}$ from a point on $\ell(t)$ $((3,2,-1))$ to the point $(3,3,1)$. $\vec{v} = (0,1,2)$, $\vec{d} = (1,-5,0)$.

$$\vec{n} = \vec{v} \times \vec{d} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & -5 & 0 \end{vmatrix} = (1\cdot 0 - 2\cdot(-5))\,i - (0-2)\,j + (0\cdot(-5) - 1)\,k$$

$$= (10, 2, -1)$$

Answer: $(10,2,-1)\cdot(x-3, y-3, z-1) = 0$

$$10x - 30 + 2y - 6 - z + 1 = 0$$

$$10x + 2y - z - 35 = 0$$

Name: ____________________
SID: ____________________

4. (a) Find the cylidrical and spherical coordinates of the point with rectangular coordinates $(-\sqrt{3}, 0, 1)$.

cylindrical: $r = \sqrt{(-\sqrt{3})^2 + 0^2} = \sqrt{3}$, $-\sqrt{3} = r\cos\theta = \sqrt{3}\cos\theta$

$-1 = \cos\theta$, $\theta = \pi$

cylindrical: $(\sqrt{3}, \pi, 1)$

spherical: $\rho = \sqrt{(-\sqrt{3})^2 + 0^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$1 = \rho\cos\varphi = 2\cos\varphi$, $\frac{1}{2} = \cos\varphi$, $0 \le \varphi \le \pi$ $\varphi = \pi/3$

$\theta = \pi$ same as cylindrical

spherical: $(2, \pi, \pi/3)$

(b) Find the equation of $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates. (For the spherical coordinates, simplify the equation.) Describe and/or draw the surface described by these equations.

cylindrical $z = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2} = r$

cylindrical: $z = r$

Spherical: $\rho\cos\varphi = \sqrt{(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$

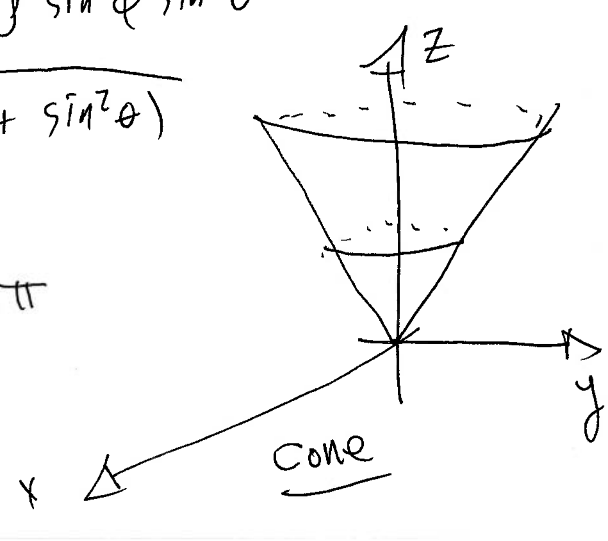
$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}$

$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi(\cos^2\theta + \sin^2\theta)}$

$\rho\cos\varphi = \rho\sin\varphi$

$\cos\varphi = \sin\varphi$, $0 \le \varphi \le \pi$

Spherical: $\varphi = \pi/4$



5. Let S be the surface defined by the equation $\sin(xy) + y^2x - 3z = 0$.

(a) Write down a function $f(x, y, z)$ of three variables and a constant c such that S is the level set of f of value c.

$f(x, y, z) = \sin(xy) + y^2x - 3z$

$c = 0$

(b) Find a real valued function $g(x, y)$ of two variables such that S is the graph of g.

$\sin(xy) + y^2x - 3z = 0$

$\sin(xy) + y^2x = 3z$

$z = \frac{\sin(xy) + y^2x}{3}$

$g(x,y) = \frac{\sin(xy) + y^2x}{3}$

Name: ________________

SID: ________________

6. (Extra Credit) Let $f(x,y,z) = x^2 - z^2 + 1$. Is there a c such that the level set of f of value c is a collection of planes? If so, what's the c and what are the planes?

$0 = x^2 - z^2$ ← ~~equate~~ the two

$0 = (x-z)(x+z)$ planes $x - z = 0$

and $x + z = 0$

$f(x,y,z) = x^2 - z^2 + 1$ has level set $0 = x^2 - z^2$

when $c = 1$: $1 = x^2 - z^2 + 1$

$0 = x^2 - z^2$

Answer: Yes, $c = 1$, the planes are the planes

$x + z = 0$ and $x - z = 0$.

Name: Solutions

SID: ____________________

20C, Ferrara, Oct 19, Version 1

Midterm 1

This exam has 7 pages and 6 problems. Make sure that your exam has all 7 pages and that your name is on every page.
Put your name and student ID on every page.
You must show your work and justify your answers to receive full credit unless otherwise stated.
If you need more space, use the pack of the pages; clearly indicate when you have done this.
You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

1	
2	
3	
4	
5	
6	
7	
Total	

20C, Ferrara, Oct 19, Version 1

Name: ______________

SID: ______________

1. Let $\mathbf{v} = (-1, 1, 4)$, $\mathbf{w} = (2, 4, 4)$, $\alpha = 7$. Calculate the following (use the space at the bottom of the page for your work):

   (a) $\mathbf{v} + \mathbf{w} =$ (1,5,8)

   (b) $\alpha\mathbf{v} =$ (-7,7,28)

   (c) $\mathbf{v} \cdot \mathbf{w} =$ 18

   (d) $\|\mathbf{v}\| =$ $\sqrt{18} = 3\sqrt{2}$

   (e) $\|\mathbf{w} =$ 6 ← TYPO

   (f) $\mathbf{v} \times \mathbf{w} =$ (-12,12,-6)

   (g) $\mathbf{w} \times \mathbf{v} =$ $-\vec{v} \times \vec{w} = (12,-12,6)$

   (h) $v \cdot (\mathbf{v} \times \mathbf{w}) =$ 0 ← TYPO

   (i) $w \cdot (\mathbf{v} \times \mathbf{w}) =$ 0

   } cross product is orthogonal to original vectors

   (j) Angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/4$.

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$

$18 = 3\sqrt{2} \cdot 6 \cos\theta$

$\cos\theta = \frac{18}{18\sqrt{2}}$

$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \cos\theta, \quad \theta = \pi/4$

$\vec{v} + \vec{w} = (-1,1,4) + (2,4,4) = (1,5,8)$

$\alpha\vec{v} = 7 \cdot (-1,1,4) = (-7,7,28)$

$\vec{v} \cdot \vec{w} = (-1,1,4) \cdot (2,4,4) = -2 + 4 + 16 = 18$

$\|\vec{v}\| = \sqrt{(-1)^2 + 1^2 + 4^2} = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

$\|\vec{w}\| = \sqrt{2^2 + 4^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ -1 & 1 & 4 \\ 2 & 4 & 4 \end{vmatrix} = (4-16)i - (-4-8)j + (-4-2)k$$

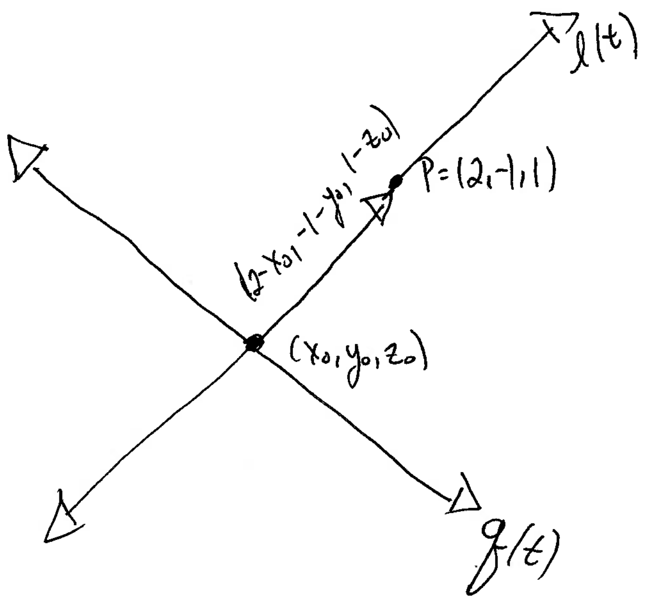
$= -12i + 12j - 6k$

$= (-12, 12, -6)$

Name: ______________________

SID: ______________________

2. Find the equation of the line that contains the point $(2, -1, 1)$, and is perpendicular and intersecting the line with equation $q(t) = (4, 3, 3) + t(3, 1, 0)$.



$l(t) = (x_0, y_0, z_0) + t(2-x_0, -1-y_0, 1-z_0)$

perpendicular:

$(2-x_0, -1-y_0, 1-z_0) \bullet (3,1,0) = 0$

$3(2-x_0) + 1\cdot(-1-y_0) + 0\cdot(1-z_0) = 0$

$6 - 3x_0 - 1 - y_0 = 0$

$5 - 3x_0 = y_0$

intersecting: $q(t_0) = (x_0, y_0, z_0)$ for some t_0

$(4,3,3) + t_0(3,1,0) = (x_0, y_0, z_0)$.

$4 + 3t_0 = x_0$

$3 + t_0 = y_0$

$3 = z_0$

plug $5 - 3x_0 = y_0$ into $3 + t_0 = y_0$ get $5 - 3x_0 = 3 + t_0$

$2 - 3x_0 = t_0$

plug $t_0 = 2 - 3x_0$ into $4 + 3t_0 = x_0$ get $4 + 3(2 - 3x_0) = x_0$

$4 + 6 - 9x_0 = x_0$

$10 = 10x_0$

$1 = x_0$

$y_0 = 5 - 3x_0 = 5 - 3\cdot 1 = 2$

$x_0 = 1,\ y_0 = 2,\ z_0 = 3$

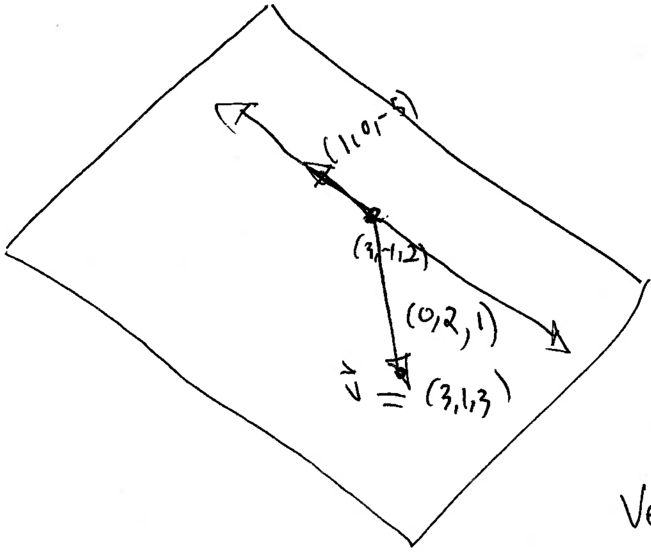
Answer: $l(t) = (1,2,3) + t(1,-3,-2)$

Name: ____________________

SID: ____________________

3. Find the equation of the plane contianing the line $\ell(t) = (3+t, -1, 2-5t)$ and the point $(3,1,3)$.

$\ell(t) = (3,-1,2) + t(1,0,-5)$



To get normal cross the direction vector for $\ell(t)$ $\big((1,0,-5)\big)$ with a vector from a point on $\ell(t)$ $\big((3,-1,2)\big)$ to $(3,1,3)$.

Vector from $(3,-1,2)$ to $(3,1,3)$ is

$\vec{v} = (3-3, 1--1, 3-2) = (0,2,1)$

Let $\vec{d} = (1,0,-5)$

$$\vec{n} = \vec{v}\times\vec{d} = \begin{vmatrix} i & j & k \\ 0 & 2 & 1 \\ 1 & 0 & -5 \end{vmatrix} = (2\cdot(-5) - 1\cdot 0)\,i - (0\cdot(-5) - 1\cdot 1)\,j + (0\cdot 0 - 2\cdot 1)\,k$$

$= -10i + j - 2k$

$= (-10, 1, -2)$

Answer: $(-10,1,-2)\cdot(x-3, y-1, z-3) = 0$

$-10x + 30 + y - 1 - 2z + 6 = 0$

$-10x + y - 2z + 35 = 0$

Name: ____________________

SID: ____________________

4. (a) Find the cylidrical and spherical coordinates of the point with rectangular coordinates $(-\sqrt{3}, 0, 1)$.

cylindrical: $r\cos\theta = -\sqrt{3}$, $r = \sqrt{(-\sqrt{3})^2 + 0^2} = \sqrt{3}$

$r\sin\theta = 0$

$\sqrt{3}\cos\theta = -\sqrt{3}$

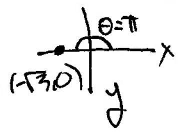
$\cos\theta = -1$

$\theta = \pi$

cylindrical: $(\sqrt{3}, \pi, 1)$

spherical: $\rho = \sqrt{(-\sqrt{3})^2 + 0^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$1 = \rho\cos\varphi = 2\cos\varphi$, $\cos\varphi = \frac{1}{2}$, $\varphi = \frac{\pi}{3}$, $\theta = \pi$



spherical: $(2, \pi, \pi/3)$

(b) Find the equation of $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates. (For the spherical coordinates, simplify the equation.) Describe and/or draw the surface described by these equations.

cylindrical $z = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2} = r$

cylindrical: $z = r$

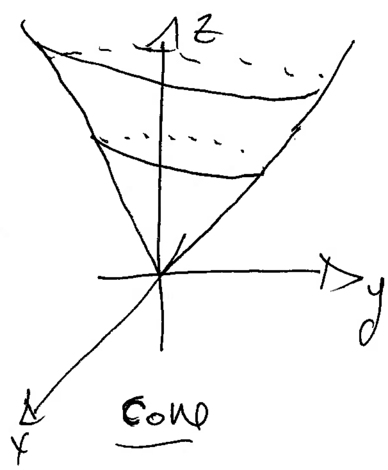
spherical: $\rho\cos\varphi = \sqrt{(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$

$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi(\cos^2\theta + \sin^2\theta)}$

$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi} = \rho\sin\varphi$

$\cos\varphi = \sin\varphi$, $0 \le \varphi \le \pi$

spherical: $\varphi = \pi/4$



5. Let S be the surface defined by the equation $\cos(xy) + x^2y - 2z = 0$.

(a) Write down a function $f(x, y, z)$ of three variables and a constant c such that S is the level set of f of value c.

$f(x,y,z) = \cos(xy) + x^2y - 2z$

$c = 0$

(b) Find a real valued function $g(x, y)$ of two variables such that S is the graph of g.

$\cos(xy) + x^2y - 2z = 0$

$\cos(xy) + x^2y = 2z$

$z = \dfrac{\cos(xy) + x^2y}{2}$

$g(x,y) = \dfrac{\cos(xy) + x^2y}{2}$

Name: __________________
SID: __________________

6. (Extra Credit) Let $f(x, y, z) = x^2 - z^2 + 1$. Is there a c such that the level set of f of value c is a collection of planes? If so, what's the c and what are the planes?

$0 = x^2 - z^2$

$0 = (x - z)(x + z)$ two planes $x = z$ and $x = -z$

$x - z = 0$ $x + z = 0$

$0 = (x - z)(x + z)$ is the level set of $f(x, y, z) = x^2 - z^2 + 1$ of level $c = 1$

$1 = x^2 - z^2 + 1$

$0 = x^2 - z^2$

Answer: Yes, $c = 1$, planes are $x - z = 0$ and $x + z = 0$