32.

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) If $(x,y) \neq (0,0)$ calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)\left(y(x^2-y^2)+2x^2y\right) - xy(x^2-y^2)\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y(x^2-y^2)(x^2+y^2) + 2x^2y(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^2-y^2)\left(x^2+y^2-2x^2\right) + 2x^2y(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^2-y^2)(y^2-x^2) + 2x^2y(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x} = \frac{2x^2y(x^2+y^2) - y(x^2-y^2)^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(x(x^2-y^2) - 2y^2x) - xy(x^2-y^2)\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{x(x^2+y^2)(x^2-y^2) - 2y^2x(x^2+y^2) - 2y^2x(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{x(x^2-y^2)(x^2+y^2-2y^2) - 2y^2x(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x(x^2-y^2)(x^2-y^2) - 2y^2x(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x(x^2-y^2)^2 - 2y^2x(x^2+y^2)}{(x^2+y^2)^2}$$

(b) Show that $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$

CANNOT plug $(0,0)$ into formula from (a) since would get $\frac{0}{0}$. Therefore use limit definition of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Note by definition of f, $f(0,0) = 0$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h \cdot 0(h^2 - 0^2)}{h^2 + 0^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{0}{h^3} = \lim_{h \to 0} 0 = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0 \cdot h(0^2 - h^2)}{0^2 + h^2} - 0}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0$$

(c) Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$, $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$

~~(a)~~ Again if you differentiate the answer from part (a), then plug in (0,0), you will get $\frac{0}{0}$ somewhere. Therefore use the limit definition of partial derivative and the answers from part (a) and (b):

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)$$

$$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(0+h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$\frac{\partial f}{\partial y}(0+h,0) = \frac{\partial f}{\partial y}(h,0) = \frac{h(h^2-0^2)^2 - 2\cdot 0^2 \cdot h(h^2-0^2)}{(h^2+0^2)^2} = \frac{h^5}{h^4} = h$$

by formula in (a)

$$\frac{\partial f}{\partial y}(0,0) = 0 \quad \text{by (b)}$$

Then

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(0+h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = \lim_{h \to 0} 1 = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$$

$$= \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0, 0+h) - \frac{\partial f}{\partial x}(0,0)}{h}$$

$$\frac{\partial f}{\partial x}(0, 0+h) = \frac{\partial f}{\partial x}(0, h) = \frac{2 \cdot 0^2 h (0^2 + h^2) - h(0^2 - h^2)^2}{(0^2 + h^2)^2} = \frac{-h^5}{h^4} = -h$$

by formula in (a)

$$\frac{\partial f}{\partial x}(0,0) = 0 \text{ by (b).}$$

Then

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0,0+h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = \lim_{h \to 0} -1 = -1$$

(d) Since the mixed partials are different at $(0,0)$, the functions $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ must not be continuous at $(0,0)$. This follows from Theorem 1 on page 151.