Homework 2 Solutions

22. Two vectors in $\mathbb{R}^3$, say $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are orthogonal if $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3 = 0$.

For part a), we solve the equation:

$$\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} b \\ 1 \end{bmatrix} = 0 \Rightarrow 2(-3) + 2b = 0$$

$$\Rightarrow b = 3$$

For part b), notice that

$$\begin{bmatrix} 2 & 0 \\ b & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

for any real number $b \in \mathbb{R}$.

+1 for trying to do dot product in a)
+1.5 for correct answer to a)
+1 for trying to do dot product in b)
+1.5 for correct answer to b).
This problem isn’t inherently hard, but there are a lot of equations. First, since the two lines intersect, we know there are real numbers \( s, t \in \mathbb{R} \) so that

\[
\begin{bmatrix}
3 \\
1 + b \\
-2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
=
\begin{bmatrix}
-1 \\
-2 \\
-1
\end{bmatrix}
\begin{bmatrix}
t \\
t \\
t
\end{bmatrix}
\]

line whose equation we don’t know yet

\[
\begin{bmatrix}
4 \\
3 + b \\
-1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
=
\begin{bmatrix}
t \\
t \\
t
\end{bmatrix}
\]

Rearrange:

Two vectors are equal, so

\[ t = 4 + as = 3 + bs = -1 + cs \ldots (\star) \]

Since the two lines are perpendicular, \([\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}] = 0 \Rightarrow a + b + c = 0 \]

Thus \((\star)\) can be written as

\[ 4 + as = 3 + bs = -1 - as - bs \]

So, \[ 3 + bs = -1 - as - bs \Rightarrow -4 - 2bs = as \]

We have \[ 4 + as = 3 + bs, \] but since \[ as = -4 - 2bs, \]

\[ 4 + (-4 - 2bs) = 3 + bs \]

\[ \Rightarrow -3bs = 3 \Rightarrow s = -\frac{1}{b} \]
We have a value for $s$, so we know

\[
\begin{bmatrix}
4 \\
3 \\
-1
\end{bmatrix} + \begin{bmatrix}
a \\ b \\ c
\end{bmatrix} \left( -\frac{1}{b} \right) = \begin{bmatrix}
t \\ t \\ t
\end{bmatrix} \Rightarrow \begin{bmatrix}
4 - \frac{a}{b} \\ 2 \\ -1 - \frac{c}{b}
\end{bmatrix} = \begin{bmatrix}
t \\ t \\ t
\end{bmatrix}
\]

Thus $4 - \frac{a}{b} = t = 2 \Rightarrow 4 - \frac{a}{b} = 2$

$\Rightarrow -\frac{a}{b} = -2$

$\Rightarrow \frac{a}{b} = 2 \Rightarrow a = 2b$

We know from before $a + b + c = 0 \Rightarrow c = -(a + b) \Rightarrow c = -3b$

Thus

\[
\begin{bmatrix}
4 \\ 3 \\ -1
\end{bmatrix} + \begin{bmatrix}
2b \\ b \\ -3b
\end{bmatrix} \left( -\frac{1}{b} \right) = \begin{bmatrix}
2 \\ 2 \\ 2
\end{bmatrix} \Rightarrow t = 2
\]

If we plug $t = 2$ into the original line equation,

\[
\begin{bmatrix}
-1 \\ -2 \\ -1
\end{bmatrix} + \begin{bmatrix}
1 \\ -2 \\ 1
\end{bmatrix} \cdot 2 = \begin{bmatrix}
0 \\ 0 \\ 0
\end{bmatrix}, \text{ so } \begin{bmatrix}
1 \\ 0 \\ 0
\end{bmatrix} \text{ is the point of intersection.}
\]

We put everything together and see that for one particular value of $b, s$,

\[
\begin{bmatrix}
3 \\ 1 \\ -2
\end{bmatrix} + \begin{bmatrix}
2 \\ 1 \\ -3
\end{bmatrix} (b, s) = \begin{bmatrix}
1 \\ 0 \\ 1
\end{bmatrix}
\]

Thus the line has direction $\begin{bmatrix}
\frac{2}{3} \\ 1 \\ -3
\end{bmatrix}$

We conclude that the line equation is

\[
\ell(t) = \begin{bmatrix}
-3 \\ 1 \\ -2
\end{bmatrix} + \begin{bmatrix}
2 \\ 1 \\ -3
\end{bmatrix} t
\]

where $t$ is a real-valued parameter.
First, a parametrized line in $\mathbb{R}^3$ is completely determined by a point it passes through and its direction vector, so we know that the line we want looks like

$$l(t) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} t$$

The line is perpendicular to $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, so

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b/2 \\ c \end{bmatrix} = 0 \Rightarrow 2a - b/2 + c = 0$$

$$\Rightarrow b = 2a + c$$

So our line now looks like $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} a \\ 2a+c \\ c \end{bmatrix} t$

Notice that if we have a plane $Ax + By + Cz = D$, then $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ is the vector perpendicular to the plane.

Thus, $\begin{bmatrix} a \\ 2a+c \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} = 0 \Rightarrow a = \frac{3}{4} c$

So our line is now $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} c \\ \frac{5}{2} c \\ c \end{bmatrix} t$

Since $t$ ranges over all real numbers, it doesn't really matter what we choose for $c$. Let's choose $c = 4$. Then,

$$l(t) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 10 \\ 4 \end{bmatrix} t$$

(as long as $c \neq 0$)

+1 correct final answer.
Here's a rough picture of the situation:

(Not drawn to scale or with right coordinates)

To form a plane we need two vectors. One of them is already given to us in the line equation — \((1, 0, 5)\).

To find another vector, consider the vector going from \((1, 2, -1)\) to \((1, 0, 1)\). To find what this vector is, we subtract the coordinates of the points:

\[
(1, 0, 1) - (1, 2, -1) = (0, -2, 2).
\]

Given the two vectors we can find the vector that is perpendicular to both of them, and thus perpendicular to the plane. We do this with the cross product.

\[
\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \times \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -2 \end{bmatrix} + \text{3 correct normal vector}
\]

Thus the plane equation takes the form \(10x - 2y - 2z = D\). But we know \((1, 0, 1)\) is on the plane. Thus \(10 \cdot 1 - 2 \cdot 0 - 2 \cdot 1 = 8\).

So, the plane equation is

\[
10x - 2y - 2z = 8
\]

\[
\Rightarrow 5x - y - z = 4 \quad + \text{2 correct final equation}
\]