Homework 2 Solutions

(22) Two vectors in $\mathbb{R}^3$, say $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

are orthogonal if $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$.

For part a), we solve the equation:

$$\begin{bmatrix} 2 \\ b \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 0 \Rightarrow 2(-3) + 2b = 0$$

$$\Rightarrow \boxed{b = 3}$$

For part b), notice that

$$\begin{bmatrix} 2 \\ b \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \text{for any real number } \boxed{b \in \mathbb{R}}$$

+1 for trying to do dot product in a)
+1.5 for correct answer to a)
+1 for trying to do dot product in b)
+1.5 for correct answer to b).

+1 off to a start, but not much progress
+2.5 a lot of genuine effort, but wrong method
+4 correct method, wrong final answer
+5 correct final answer

(26) This problem isn't inherently hard, but there are a lot of equations. First, since the two lines intersect, we know there are real numbers $s, t \in \mathbb{R}$ so that

$$\underbrace{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} s}_{\text{line whose equation we don't know yet}} = \underbrace{\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t}_{\text{line given in the problem}}$$

- Rearrange: $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} s = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$

- Two vectors are equal, so

$$t = 4 + as = 3 + bs = -1 + cs \quad \ldots \quad (*)$$

- Since the two lines are perpendicular, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow a + b + c = 0$

- Thus $(*)$ can be written as

$$4 + as = 3 + bs = -1 - as - bs$$

- So, $3 + bs = -1 - as - bs \Rightarrow -4 - 2bs = as$

- We have $4 + as = 3 + bs$, but since $as = -4 - 2bs$,

$$4 + (-4 - 2bs) = 3 + bs$$

$$\Rightarrow -3bs = 3 \Rightarrow s = -\frac{1}{b}$$

- We have ~~a value~~ an expression for s, so we know

$$\begin{bmatrix}4\\3\\-1\end{bmatrix}+\begin{bmatrix}a\\b\\c\end{bmatrix}\left(-\frac{1}{b}\right)=\begin{bmatrix}t\\t\\t\end{bmatrix}\Rightarrow\begin{bmatrix}4-a/b\\2\\-1-c/b\end{bmatrix}=\begin{bmatrix}t\\t\\t\end{bmatrix}$$

- Thus $4-a/b=t=2 \Rightarrow 4-a/b=2$

$$\Rightarrow -\frac{a}{b}=-2$$

$$\Rightarrow \frac{a}{b}=2 \Rightarrow a=2b$$

- We know from before $a+b+c=0 \Rightarrow c=-(a+b) \Rightarrow c=-3b$

- Thus $\begin{bmatrix}4\\3\\-1\end{bmatrix}+\begin{bmatrix}2b\\b\\-3b\end{bmatrix}\left(-1/b\right)=\begin{bmatrix}2\\2\\2\end{bmatrix} \Rightarrow t=2$

- If we plug $t=2$ into the original line equation,

$$\begin{bmatrix}-1\\-2\\-1\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot 2=\begin{bmatrix}1\\0\\1\end{bmatrix},\ \text{so}\ \begin{bmatrix}1\\0\\1\end{bmatrix}\ \text{is the point of intersection.}$$

- We put everything together and see that for one particular value of $b \cdot s$,

$$\begin{bmatrix}3\\1\\-2\end{bmatrix}+\begin{bmatrix}2\\1\\-3\end{bmatrix}(bs)=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Thus the line has direction $\begin{bmatrix}2\\1\\-3\end{bmatrix}$

- We conclude that the line equation is $\boxed{l(t)=\begin{bmatrix}3\\1\\-2\end{bmatrix}+\begin{bmatrix}2\\1\\-3\end{bmatrix}t}$

where t is a real-valued parameter.

(32) First, a parametrized line in $\mathbb{R}^3$ is completely determined by a point it passes through and its direction vector, so we know that the line we want looks like

$$\ell(t) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} t$$

The line is perpendicular to $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, so

+2 for using this information

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow 2a - b + c = 0$$

$$\Rightarrow b = 2a + c$$

So our line now looks like $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} a \\ 2a+c \\ c \end{bmatrix} t$

Notice that if we have a plane $Ax + By + Cz = D$, then $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ is the vector perpendicular to the plane.

+2 for using this information

Thus, $\begin{bmatrix} a \\ 2a+c \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = 0 \xRightarrow[\text{solve}]{} a = 3/4\, c$

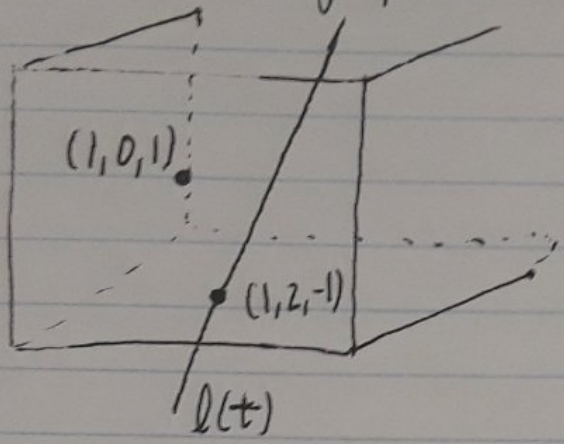
So our line is now $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 3/4\, c \\ 5/2\, c \\ c \end{bmatrix} t$

Since t ranges over all real numbers, it doesn't really matter what we choose for c. Let's choose $c = 4$. Then,
(as long as $c \neq 0$)

$$\boxed{\ell(t) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 10 \\ 4 \end{bmatrix} t .}$$

+1 correct final answer

(33) Here's a rough picture of the situation:



(Not drawn to scale or w/ right coordinates)

To form a plane we need two vectors. One of them is already given to us in the line equation — $(1, 0, 5)$.

To find another vector, consider the vector going from $(1, 2, -1)$ to $(1, 0, 1)$. To find what this vector is, we subtract the coordinates of the points:

$$(1,0,1) - (1,2,-1) = (0,-2,2).$$

Given the two vectors we can find the vector that is perpendicular to both of them, and thus perpendicular to the plane. We do this with the cross product.

$$\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \times \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -2 \end{bmatrix}$$

+3 correct normal vector

Thus the plane equation takes the form $10x - 2y - 2z = D$.
But we know $(1,0,1)$ is on the plane.
Thus $10 \cdot 1 - 2 \cdot 0 - 2 \cdot 1 = 8$.
So, the plane equation is

$$10x - 2y - 2z = 8$$

+2 correct final equation

$$\Rightarrow \boxed{5x - y - z = 4}$$