Rmk Each problem is out of 5 pts, but the first three will be scaled to $6\frac{2}{3} \approx 6.66$ points each.

Homework 6 Solutions

3.3 #28

Solution 1: We want to minimize the distance $\sqrt{x^2+y^2+z^2}$, where (x, y, z) is a point on the plane $2x-y+2z=20$. Observe that ① $y=2x+2z-20$ and that ② $0 \le \sqrt{x^2+y^2+z^2}$

Some function to minimize +2

By ① and ②, it is enough to minimize

$$f(x,y,z) = x^2+y^2+z^2 = x^2+(2x+2z-20)^2+z^2$$

Redefine the function: $g(x,z) = x^2+(2x+2z-20)^2+z^2$

Correct system of eqns w/ partial derivatives +2

Now $\frac{\partial g}{\partial x} = 2x+4(2x+2z-20) = 10x+8z-80$

$$\frac{\partial g}{\partial z} = 2z+4(2x+2z-20) = 8x+10z-80$$

You can check that solving $\begin{cases} \partial g/\partial x\,(x,z) = 0 \\ \partial g/\partial z\,(x,z) = 0 \end{cases}$

gives $x = z = \frac{40}{9}$. By ①, $y = 2(40/9)+2(40/9)-20 = -20/9$. Clearly, $(x,y,z) = (40/9, -20/9, 40/9)$ does not maximize the distance function, since $g(x,z)$ is unbounded. Thus we have found a point for which the distance is minimal.

$$\boxed{(x,y,z) = (40/9, -20/9, 40/9)} \quad +1$$

Solution 2: With some geometric intuition, we can just find the normal vector that starts from the origin and ends on a point on the plane. This vector has to be in the form: +2

$$\vec{v} = (0,0,0) + \underbrace{(2,-1,2)}_{\text{normal vector of plane}}t = (2,-1,2)t = \underline{(2t,-t,2t)}$$

Since it is on the plane, $2(2t)-(-t)+2(2t) = 20 \Rightarrow \underline{t = 20/9}$ +2

Thus $(2t, -t, 2t) = \underline{\left(\frac{40}{9}, -\frac{20}{9}, \frac{40}{9}\right)}$ +1

3.3 #44

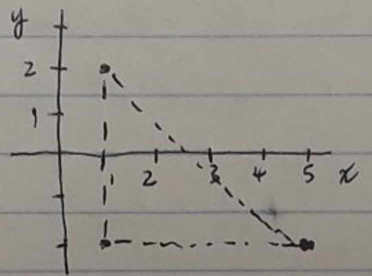
Define $f(x,y) = 1 + xy + x - 2y$

Check this (+1)

First, by solving for critical points:

$$\begin{cases} \partial/\partial x\, f(x,y) = 0 \\ \partial/\partial y\, f(x,y) = 0 \end{cases} \Rightarrow \begin{cases} y+1=0 \\ x-2=0 \end{cases} \Rightarrow (x,y) = (2,-1)$$ which is in the triangle

Notice that $f(2,-1) = 3$, but we haven't checked the boundary of the triangle yet. Here's what the triangle looks like:



(+1)

- Along the vertical line, we have $f(x,y) = f(1,y) = 2-y$ where $-2 \le y \le 2$. Clearly the extreme values occur at the endpoints $y = \pm 2$. $\Rightarrow (x,y) = (1,2)$ or $(1,-2)$

(+1)

- Along the horizontal line, we similarly have $f(x,y) = f(x,-2) = 5-x$ where $1 \le x \le 5$. $\Rightarrow (x,y) = (1,-2)$ or $(5,-2)$

(+1)

- Finally along the diagonal, we have $f(x,y) = f(x, 3-x) = -x^2+6x-5$ where $1 \le x \le 5 \Rightarrow (x,y) = (3,0)$, $(1,2)$, or $(5,-2)$

Now putting everything together, we check: $f(1,2) = 0$ $f(5,-2) = 0$ $f(1,-2) = 4$ $f(3,0) = 4$

Thus the critical point we found in the beginning doesn't matter, and that absolute max of f on $D = 4$ at $(1,-2)$ and $(3,0)$ $\boxed{+0.5}$

absolute min of f on $D = 0$ at $(1,2)$ and $(\cancel{-5},2)$ $\boxed{+0.5}$

typo: $(5,-2)$

3.4 #13

On the unit circle $\{(x,y) \mid x^2+y^2=1\}$, f has max/min, since the unit circle is closed and bounded.

If $f(x,y)=x^2+y^2+xy$, the constraint $g(x,y)=x^2+y^2-1=0$, then we have, using Lagrange Multipliers

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \Rightarrow \left(2x+y, x+2y\right) = \lambda\left(2x, 2y\right) \quad +1$$

$$\Rightarrow \begin{cases} 2x+y=2\lambda x \\ 2y+x=2\lambda y \end{cases} \Rightarrow \begin{cases} x=\frac{y}{2(\lambda-1)} \\ y=\frac{x}{2(\lambda-1)} \end{cases} \Rightarrow x=\frac{x}{4(\lambda-1)^2} \Rightarrow 1=\frac{1}{4(\lambda-1)^2}$$

$\Rightarrow \lambda=\frac{1}{2}, \frac{3}{2} \Rightarrow$ either $x=y$ or $x=-y$ (note that we need $x^2+y^2=1$)

+2

$$\Rightarrow (x,y) = \left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right), \left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right), \left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right), \left(-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right)$$

$$f\left(\pm\tfrac{\sqrt{2}}{2}, \pm\tfrac{\sqrt{2}}{2}\right) = \tfrac{3}{2}, \quad f\left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right) = f\left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right) = \tfrac{1}{2}$$

With similar reasoning, if our domain is $\{(x,y) \mid x^2+y^2=r\}$, the max value for a fixed $0<r\leq 1$ occurs on the path $x=y$. (min value: $x=-y$).

Thus on the entire unit circle (interior + boundary), the max value occurs when the radius is as large as possible ($r=1$) and $x=y$. Thus the max value of f on the unit circle is $\boxed{3/2}$ +1

It's tempting to say that the min value is $\frac{1}{2}$, but that's not true. Notice that $x^2+y^2+xy \geq 0$ for all $x,y \in \mathbb{R}$ (why?), and that if $(x,y)=(0,0)$, $f(0,0)=0$. Thus the min value of f on the unit circle is $\boxed{0}$. +1

Extra Credit - all or nothing!

3.4 (14) The max and min of f do not exist. For example, I can take $x = 500{,}000{,}000$, $y = 500{,}000{,}000$ and $z = 1 - x - y$. Then obviously $x + y + z = 1$, but that $f(500{,}000{,}000, 500{,}000{,}000) = 2 \cdot 5 \cdot 10^8 + 5 \cdot 10^8$ = very large number. Actually, we can repeat the above logic for arbitrarily large x and y, so the max does not exist.

Repeating everything above with very large negative numbers, say $x = -10^{12}$, $y = -10^{812}$ (I just made up large numbers), we see that the min does not exist.

In serious mathematics, though, you should try to show that for every $M > 0$ there exists $x, y \in \mathbb{R}$ such that $2x + y > M$, if you want to show that there is no max.

(You could also do this problem by showing that there are no solutions by using Lagrange Multipliers)

Reasonable argument that shows BOTH why the max/min do not exist +5

3.4 (15) Apply Lagrange Multipliers to $f(x,y) = 4x + 2y$
with constraint $g(x,y) = 2x^2 + 3y^2 - 21 = 0$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$$

$$(4, 2) = \lambda(4x, 6y)$$

$$\begin{cases} x \cdot \lambda = 1 \\ 3y \cdot \lambda = 1 \end{cases} \Rightarrow \begin{array}{l} x = 1/\lambda \\ y = 1/3\lambda \end{array} \Rightarrow 2\left(\frac{1}{\lambda}\right)^2 + 3\left(\frac{1}{3\lambda}\right)^2 - 21 = 0$$

$$\Rightarrow \lambda = \pm 1/3 \Rightarrow x = \frac{1}{\pm 1/3} = \pm 3 \Rightarrow y = \pm 1$$

$$\Rightarrow (x,y) = (3,1) \text{ or } (-3,-1)$$

$$f(3,1) = 14, \quad f(-3,-1) = -14$$

Thus max value of f $= 14$, min value of f $= -14$.

Correct max/min : +5 (of course need to show work)