Homework 7

9.

The acceleration vector is;

$$\mathbf{a} = \mathbf{c}''(t).$$

Therefore, calculate the derivatives of (1).

First, find $\mathbf{c}'(t)$ by differentiating $\mathbf{c}(t)$;

$$\mathbf{c}'(t) = (-a\sin t, a\cos t, b)$$

Now, finding the double differential;

$$\begin{aligned}\mathbf{a} &= \mathbf{c}''(t) \\ &= (-a\cos t, -a\sin t, 0)\end{aligned}$$

In the above, the *z*-coordinate of acceleration vector is always zero, therefore the acceleration vector lies on the plane $z = 0$ in the *xy*-plane. **Hence, acceleration vector is parallel to *xy*-plane.**

+3 if he gives correct double differential.

+2 if he gives good explain.

11.

(a)

Consider the following path

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$

Differentiate with respect to *t*,

$$\mathbf{c}'(t) = (-\sin t, \cos t, 1)$$

Since *z*-coordinate of $\mathbf{c}'(t)$ is 1, therefore $\mathbf{c}'(t) \neq 0$ for all *t*.

Thus, the path $\mathbf{c}(t)$ is regular.

+1.5

(b)

Consider the path

$$\mathbf{c}(t) = (t^3, t^5, \cos t)$$

Differentiate with respect to *t*,

$$\mathbf{c}'(t) = (3t^2, 5t^4, \sin t)$$

For, $t = 0$,

$$\mathbf{c}'(0) = (3(0)^2, 5(0)^4, \sin 0)$$
$$= (0, 0, 0)$$

Therefore the path $\mathbf{c}(t)$ is not regular at 0.

Thus, the path $\mathbf{c}(t)$ is not regular.

+1.5

(c)

Consider the following path

$$\mathbf{c}(t) = (t^2, e^t, 3t + 1)$$

Differentiate with respect to *t*,

$$\mathbf{c}'(t) = (2t, e^t, 3)$$

Since *z*-coordinate of $\mathbf{c}'(t)$ is 3, therefore $\mathbf{c}'(t) \neq 0$ for all *t*.

Thus the path $\mathbf{c}(t)$ is regular.

+1.5

+0.5 if he wrote something for this question.

20.

Differentiate with respect t either side.

$$\frac{d}{dt}\left(\|\mathbf{r}(t)\|^2\right)=\frac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}(t)\right)$$

Since $\|\mathbf{r}(t)\|$ is a constant function and derivative is zero, so that,

$$0=\frac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}(t)\right)$$

$$0=\mathbf{r}(t)'\cdot\mathbf{r}(t)+\mathbf{r}(t)\cdot\mathbf{r}'(t) \qquad [\text{Dot product rule of differentiation}]$$

$$0=2\mathbf{r}(t)'\cdot\mathbf{r}(t)$$

$$0=\mathbf{r}(t)'\cdot\mathbf{r}(t)$$

This yields the result, $\mathbf{r}'(t)$ that is perpendicular to $\mathbf{r}(t)$.

+5 for any reasonable proof.

13.

Consider the path,

$$c(t)=\left(2t,t^2,\log t\right):t>0$$

The objective is to find the arc length of c between the points $(2,1,0)$ and $(4,4,\log 2)$.

The arc length of a path $c(t)$ is given by,

$$L(c)=\int_{t_0}^{t_1}\|c'(t)\|\,dt$$

$$=\int_{t_0}^{t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt \qquad \text{.......}(1)$$

Where, $[t_0,t_1]$ is the interval of time t.

Differentiate function with respect to t.

$$c'(t) = \frac{d}{dt}\left(2t, t^2, \log t\right)$$
$$= \left(\frac{d}{dt}(2t), \frac{d}{dt}(t^2), \frac{d}{dt}(\log t)\right)$$
$$= \left(2\frac{d}{dt}(t), \frac{d}{dt}(t^2), \frac{d}{dt}(\log t)\right)$$
$$= \left(2(1), 2t^{2-1}, \frac{1}{t}\right) \quad \text{Use } \frac{d}{dx}(x^n) = nx^{n-1}, \frac{d}{dx}\log(x) = \frac{1}{x}$$
$$= \left(2, 2t, \frac{1}{t}\right)$$

Find the limits as follows:

At $t = 1$,

$$c(1) = \left(2(1), (1)^2, \log(1)\right)$$
$$= \left(2, 1, \log(1)\right)$$
$$= (2, 1, 0)$$

At $t = 2$,

$$c(2) = \left(2(2), (2)^2, \log(2)\right)$$
$$= \left(4, 4, \log(2)\right)$$

Thus, the required limits $\{1 \le t \le 2\}$.

Substitute these values in equation (1).

$$L=\int_1^2\sqrt{(2)^2+(2t)^2+\left(\frac{1}{t}\right)^2}\,dt \quad \text{Substitute } c'(t)=\left(2,2t,\frac{1}{t}\right)$$

$$=\int_1^2\sqrt{4+4t^2+\frac{1}{t^2}}\,dt$$

$$=\int_1^2\sqrt{\frac{4t^2+4t^4+1}{t^2}}\,dt$$

$$=\int_1^2\sqrt{\frac{4t^4+4t^2+1}{t^2}}\,dt$$

$$=\int_1^2\sqrt{\frac{(2t^2)^2+2\cdot 2t\cdot 1+(1)^2}{t^2}}\,dt \quad \text{Rewrite the term}$$

$$=\int_1^2\sqrt{\frac{(2t^2+1)^2}{t^2}}\,dt \quad \text{Use } (a+b)^2=a^2+2ab+b^2$$

$$=\int_1^2\sqrt{\left(\frac{2t^2+1}{t}\right)^2}\,dt$$

$$=\int_1^2\left(\frac{2t^2+1}{t}\right)dt$$

$$=\int_1^2\left(\frac{2t^2}{t}+\frac{1}{t}\right)dt$$

Continue the above step,

$$=\int_1^2\left(2t+\frac{1}{t}\right)dt$$

$$=\left[2\left(\frac{t^2}{2}\right)+\ln(t)\right]_1^2 \quad \text{Use } \int x^n dx=\frac{x^{n+1}}{n+1},\int\frac{1}{x}dx=\ln(x)$$

$$=\left[t^2+\ln(t)\right]_1^2$$

$$=\left[\left(2^2+\ln(2)\right)-\left(1^2+\ln(1)\right)\right]$$

$$=\left[\left(4+\ln(2)\right)-(1+0)\right] \quad \text{Since } \log(1)=0$$

$$=\left[4+\ln(2)-1\right]$$

$$=\ln(2)+3$$

Thus, the required arc length is $\boxed{3+\log(2)}$.