MATH 20C HW8 Solutions

Good luck on finals!

5.2 #28 The most important part of this problem is knowing the region of integration; it's not too important to know what the graph of $z = x^2 + y^4$ looks like. From the problem description, we see that we want to find the volume bounded by the function $z = x^2 + y^4$, in the first octant, and on the rectangle $\{(x, y) \in R^2 \mid (x, y) \in [0, 1] \times [0, 1]\}$. This leads to:

$$\int_0^1 \left(\int_0^1 x^2 + y^4 dx \right) dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + xy^4 \right) \Bigg|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^4 \right) dy$$

$$= \left(\frac{1}{3} y + \frac{1}{5} y^5 \right) \Bigg|_{y=0}^{y=1}$$

$$= \frac{1}{3} + \frac{1}{5} = \boxed{\frac{8}{15}}$$

5.3 #4 Note: I will leave out the calculations once the integral becomes Math 20AB level. To see what the regions of integration look like you can go on Desmos and type in one line (for example part (a))

$$0 \le x \le y^2 \ \ \{-3 \le y \le 2\}$$

(a) The region of integration is the set of all (x, y) such that $-3 \le y \le 2$ and that $0 \le x \le y^2$. We directly calculate that:

$$\int_{-3}^2 \left(\int_0^{y^2} x^2 + y dx \right) dy = \int_{-3}^2 \left(\frac{1}{3}(y^2)^3 + (y^2)y \right) dy$$

$$= \int_{-3}^2 \left(\frac{1}{3} y^6 + y^3 \right) dy = \left(\frac{1}{3} \left(\frac{1}{7} y^7 \right) + \frac{1}{4} y^4 \right) \Bigg|_{y=-3}^{y=2} = \boxed{\frac{7895}{84}}$$

(b) The region of integration is the set of all (x, y) such that $-1 \leq x \leq 1$ and that $-2|x| \leq y \leq |x|$. We split the original double integral into two different parts, depending on if x is positive or negative. Notice that $|x| = x$ if x is positive and $|x| = -x$ if x is negative. We get:

$$\int_0^1 \int_{-2x}^{x} e^{x+y} dydx + \int_{-1}^{0} \int_{2x}^{-x} e^{x+y} dydx$$

The rest of this problem is just routine computation. You can check that the final answer is $\boxed{\frac{3e^2 - 5 + 6e^{-1} + 2e^{-3}}{6}}$.

(c) The region of integration is the set of all (x, y) such that $0 \leq x \leq 1$ and that $0 \leq y \leq (1 - x^2)^{1/2}$. Notice that the region of integration is just the unit circle restricted to the first quadrant. Since the function being integrated over x and y is 1, the integral basically calculates the two-dimensional area of the region of integration. Alternatively, you can think of it as a 3-dimensional volume with a quarter circle as the base area and height 1 (in which case we disregard units and calculate volume = base area * height = base area). With that being said,

$$\int_0^1 \int_0^{(1-x^2)^{1/2}} dydx = \frac{1}{4}\pi R^2 = \boxed{\frac{\pi}{4}}$$

(d) The region of integration is the set of all (x, y) such that $0 \leq x \leq \pi/2$ and that $0 \leq y \leq cos(x)$. We directly calculate that:

$$\int_0^{\pi/2} \int_0^{cos(x)} ysin(x)dydx$$

$$= \int_0^{\pi/2} sin(x) \left(\frac{1}{2} y^2 \right) \Bigg|_{y=0}^{y=cos(x)} dx$$

$$= \int_0^{\pi/2} \frac{1}{2} sin(x)cos^2(x)dx = \boxed{\frac{1}{6}}$$

(e) The region of integration is the set of all (x, y) such that $0 \leq y \leq 1$ and that $y^2 \leq x \leq y$. We have:

$$\int_0^1 \int_{y^2}^{y} (x^n + y^m)dxdy$$

$$= \int_0^1 \left(\frac{x^{n+1}}{n+1} + xy^m \right) \Bigg|_{x=y^2}^{x=y} dy$$

$$= \int_0^1 \frac{y^{n+1}}{n+1} + y^{m+1} - \frac{(y^2)^{n+1}}{n+1} - (y^2)y^m dy$$

$$= \frac{1}{(n+1)(n+2)} + \frac{1}{m+2} - \frac{1}{(n+1)(2n+3)} - \frac{1}{m+3}$$

(f) The region of integration is the set of all (x, y) such that $-1 \leq x \leq 0$ and that $0 \leq y \leq 2(1-x^2)^{1/2}$. We have:

$$\int_{-1}^{0} \int_{0}^{2(1-x^2)^{1/2}} xdydx$$

$$= \int_{-1}^{0} (xy)\Big|_{y=0}^{y=2(1-x^2)^{1/2}} dx$$

$$= \int_{-1}^{0} 2x(1-x^2)^{1/2} dx = \boxed{-\frac{2}{3}}$$

5.3 #12 The region of integration is the set of all (x, y) such that $\pi \leq x \leq 2\pi$ and that $x \leq y \leq 2x$. We have:

$$\int\int_D cos(y)dxdy = \int\int_D cos(y)dydx$$

$$= \int_{\pi}^{2\pi} \int_{x}^{2x} cos(y)dydx$$

$$= \int_{\pi}^{2\pi} sin(y)\Big|_{y=x}^{y=2x} dx$$

$$= \int_{\pi}^{2\pi} (sin(2x) - sin(x))dx = \boxed{2}$$

5.4 #4

(a) We have:

$$\int_{-1}^{1} \int_{|y|}^{1} (x+y)^2 dxdy$$

$$= \int_{0}^{1} \int_{y}^{1} (x+y)^2 dxdy + \int_{-1}^{0} \int_{-y}^{1} (x+y)^2 dxdy$$

$$= \int_{0}^{1} \left(\frac{(x+y)^3}{3}\right)\Bigg|_{x=y}^{x=1} dy + \int_{0}^{1} \left(\frac{(x+y)^3}{3}\right)\Bigg|_{x=-y}^{x=1} dy = \boxed{\frac{2}{3}}$$

(b) We have:

$$\int_{-3}^{1} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 dxdy$$

$$= \int_{-3}^{1} \frac{x^3}{3}\Big|_{x=-\sqrt{9-y^2}}^{x=\sqrt{9-y^2}} dy$$

$$= \int_{-3}^{1} \frac{2}{3}(9-y^2)^{3/2} dy = \boxed{\frac{43}{3\sqrt{2}} + \frac{81\pi}{8} + \frac{81}{4} arcsin(1/3)}$$

For the last integral, evaluate it by first setting $y = 3cos(\theta)$ or $y = 3sin(\theta)$, then use the techniques for trig integrals.

(c) It's impossible to evaluate this integral unless if we change the order of integration, since e^{x^2} has no known antiderivative in terms of the elementary functions we have seen. Notice that by manipulating the inequalities, the bounds $0 \leq y \leq 4$ and $y/2 \leq x \leq 2$ imply that $0 \leq x \leq 2$ and that $0 \leq y \leq 2x$. This allows us to calculate:

$$\int_0^4 \int_{y/2}^2 e^{x^2} dxdy = \int_0^2 \int_0^{2x} e^{x^2} dydx$$

$$= \int_0^2 ye^{x^2}\Big|_{y=0}^{y=2x} dx = \int_0^2 2xe^{x^2} dx = \boxed{e^4 - 1}$$

(d) Like part (c), we must change the order of integration. If $0 \leq y \leq 1$ and $arctan(y) \leq x \leq \pi/4$, then we can manipulate these inequalities to get that $0 \leq x \leq \pi/4$ and that $0 \leq y \leq tan(x)$ (Notice that by applying the tangent function to both sides of $arctan(y) \leq x$, we get $y \leq tan(x)$). Now we have:

$$\int_0^1 \int_{arctan(y)}^{\pi/4} sec^5(x) dxdy = \int_0^{\pi/4} \int_0^{tan(x)} sec^5(x) dydx$$

$$= \int_0^{\pi/4} ysec^5(x)\Big|_{y=0}^{y=tan(x)} dx$$

$$= \int_0^{\pi/4} tan(x)sec^5(x) dx = \boxed{\tfrac{1}{5}(4\sqrt{2} - 1)}$$

5.4 #12 From the problem description, we have $x^2 \leq y \leq 10 - x^2$. The problem here is that we don't have an upper bound on x. By the previous inequality, we certainly have $x^2 \leq 10 - x^2$, which implies $2x^2 \leq 10$ and that $x \leq \sqrt{5}$ (Notice that we disregard the $-\sqrt{5}$ because the problem tells us that $x > 0$). With this last piece of information in our hands, we calculate:

$$\int_0^{\sqrt{5}} \int_{x^2}^{10-x^2} y^2\sqrt{x} dydx = \int_0^{\sqrt{5}} \left(\frac{1}{3}y^3\sqrt{x}\right)\Bigg|_{y=x^2}^{y=10-x^2} dx =$$

$$= \int_0^{\sqrt{5}} \frac{1}{3}(10-x^2)^3\sqrt{x} - \frac{1}{3}(x^2)^3\sqrt{x} dx = \boxed{\frac{78800*5^{3/4}}{693}}$$