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Midterm 1

This exam has 6 pages and 5 problems. Make sure that your exam has all 6 pages and that your name is on every page.
Put your name and student ID on every page.
You must show your work and justify your answers to receive full credit unless otherwise stated.
If you need more space, use the back of the pages; clearly indicate when you have done this.
You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

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1. (a) (6 points) Let $\mathbf{v} = (3, -2, 4)$, $\mathbf{w} = (5, -3, 5)$, $\alpha = -2$. Calculate the following (use the blank space below for your work)
   (i) $\mathbf{v} + \mathbf{w} =$ $(8, -5, 9)$
   (ii) $\alpha\mathbf{v} =$ $(-6, 4, -8)$
   (iii) $\mathbf{v} \cdot \mathbf{w} =$ $29.$
   (iv) $\|\mathbf{v}\| =$ $\sqrt{29}$
   (v) $\|\alpha\mathbf{v} + \mathbf{w}\| =$ $\sqrt{11}$

$\vec{v} + \vec{w} = (3, -2, 4) + (5, -3, 5) = (8, -5, 9)$

$\alpha\vec{v} = -2(3, -2, 4) = (-6, 4, -8)$

$\vec{v} \cdot \vec{w} = (3, -2, 4)(5, -3, 5) = 15 - 6 + 20 = 29.$

$\|\vec{v}\| = \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{9 + 4 + 16} = \sqrt{29}$

$\|\alpha\vec{v} + \vec{w}\| = \|(-6, 4, -8) + (5, -3, 5)\| = \|(-1, 1, -3)\| = \sqrt{(-1)^2 + 1^2 + (-3)^2} = \sqrt{11}$

(b) (4 points) Let $\mathbf{v}, \mathbf{w}$ be two vectors. What is $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$? Why? What is $\mathbf{w} \cdot (\mathbf{w} \times \mathbf{v})$? Why?

$\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and $\vec{w} \cdot (\vec{w} \times \vec{v}) = 0$ since $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are orthogonal to $\vec{v}$ and $\vec{w}$, and if two vectors are orthogonal, their dot product is 0.

Alternative solution: $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$. Then

$$\vec{v} \times \vec{w} = \det\begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$$

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = (v_1, v_2, v_3) \cdot (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$$

$$= v_1v_2w_3 - v_1v_3w_2 + v_2v_3w_1 - v_2v_1w_3 + v_3v_1w_2 - v_3v_2w_1 = 0$$

cancellation Similarly $\vec{w} \cdot (\vec{w} \times \vec{v}) = 0.$

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2. (a) (6 points) Find a parametric equation of the line through the points $(1, -1, 2)$ and $(2, -3, 2)$.

There are many solutions. Here is one:

direction vector: $(2,-3,2) - (1,-1,2) = (1,-2,0)$

$\ell(t) = (1,-1,2) + t\,(1,-2,0)$

$$\boxed{\ell(t) = (1+t,\ -1-2t,\ 2)}$$

(b) (4 points) Find a parametric equation, $\ell(t)$, for the line from part (a) such that $\ell(0) = (1, -1, 2)$ and the direction vector for $\ell(t)$ is a unit vector. (There are two possibilities for $\ell(t)$.)

normalize direction vector: $\dfrac{(1,-2,0)}{\|(1,-2,0)\|} = \dfrac{(1,-2,0)}{\sqrt{1^2+(-2)^2+0}} = \dfrac{1}{\sqrt{5}}(1,-2,0)$

$= \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$ ← unit vector in same direction as $(1,-2,0)$

$\ell(t) = (1,-1,2) + t\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$, $\ell(0) = (1,-1,2)$

$$\boxed{\ell(t) = \left(1+\frac{t}{\sqrt{5}},\ -1-\frac{2}{\sqrt{5}}t,\ 2\right)}$$

other possibility for $\ell(t)$: $\ell(t) = (1,-1,2) - t\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$

$\ell(t) = \left(1-\frac{t}{\sqrt{5}},\ -1+\frac{2}{\sqrt{5}}t,\ 2\right)$

3. (10 points) Find an equation for the plane containing the three points $(1,2,3)$, $(-1,3,2)$, and $(1,0,1)$.

There are many solutions. Here is one:

$(x_0, y_0, z_0) = (1,0,1)$ (could choose any of the points)

$$\left.\begin{aligned} \vec{v} &= (1,2,3) - (1,0,1) = (0,2,2) \\ \vec{w} &= (-1,3,2) - (1,0,1) = (-2,3,1) \end{aligned}\right\} \text{two vectors in plane.}$$

normal vector to plane.

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} i & j & k \\ 0 & 2 & 2 \\ -2 & 3 & 1 \end{pmatrix} = \cancel{8}$$

$$= (2\cdot 1 - 2\cdot 3)i - (0\cdot 1 - 2\cdot(-2))j + (0\cdot 3 - 2\cdot(-2))k$$

$$= -4i - \overset{4}{\cancel{2}}j + \overset{4}{\cancel{2}}k = (-4, \overset{-4}{\cancel{2}}, \overset{4}{\cancel{2}})$$

Equation for plane $\boxed{-4(x-1) - \overset{4}{\cancel{2}}(y-0) + \overset{4}{\cancel{2}}(x-1) = 0}$

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4. Let S be the set of points (x, y, z) in $\mathbb{R}^3$ satisfying $x^2+y^2+z^2=1$, so S is the sphere of radius 1 centered at the origin.

   (a) (3 points) What is a real valued function $f(x, y, z)$ and a level c, such that S is the level surface of f of level c?

S is the level ~~set~~ surface of $f(x,y,z) = x^2+y^2+z^2$ of level $c=1$.

[Once again, there are many solutions. Another one is: S is the level surface of ~~f(x,y)~~ $f(x,y,z) = x^2+y^2+z^2+27$ of level 28.)

   (b) (4 points) Is S the graph of a real valued function $g(x, y)$? Why or why not? If so, what is the $g(x, y)$ such that S is the graph of $g(x, y)$?

No S is not the graph of a function $g(x,y)$. This is because if we solve for z, we get two solutions: $z = \sqrt{1-x^2-y^2}$ and $z = -\sqrt{1-x^2+y^2}$. Therefore S does not pass the vertical line test. For any (x,y) such that ~~#~~ $x^2+y^2 \leq 1$, there are 2 z values satisfying $x^2+y^2+z^2=1$.

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5. (8 points) Find a parametric equation for the line perpendicular to the line $\mathbf{s}(t) = (1+t, -3, 2-t)$, parallel to the plane $x + 2y - z = 3$, and containing the point $(3, -3, 2)$.

There are many possible solutions.

$\vec{v}$ = direction vector for line.

Then equation for line is $\ell(t) = (3, -3, 2) + t\vec{v}$.

Need to solve for $\vec{v}$.

Given: ① $\vec{v}$ is parallel to plane. Means $\vec{v}$ is ~~p.~~ ~~or~~ perpendicular to normal vector to plane. Normal vector to plane is $\hat{n} = (1, 2, -1)$.

② $\vec{v}$ is perpendicular to line $s(t) = (1+t, -3, 2-t)$. Means $\vec{v}$ is perpendicular to the direction vectors of $s(t)$. Direction vector for $s(t)$ is $\vec{w} = (1, 0, -1)$.

Therefore we can take $\vec{v} = \vec{n} \times \vec{w} = \det\begin{pmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}$

$= (-2, 0, -2)$

Answer $\ell(t) = (3, -3, 2) + t(-2, 0, -2)$

$\boxed{\ell(t) = (3-2t, -3, 2-2t)}$