SID:__________________ Name: Solutions

Midterm 2

This exam has 7 pages and 6 problems. Make sure that your exam has all 7 pages and that your name is on every page.
Put your name and student ID on every page.
You must show your work and justify your answers to receive full credit unless otherwise stated.
If you need more space, use the pack of the pages; clearly indicate when you have done this.
You may not use books or calculators on this exam; one hand written 8.5in x 11in page (front and back) of notes is allowed.

SID:__________ Name: __________

1. (6 points) Is the function

$$f(x,y) = \begin{cases} \frac{x^2+2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$? Why or why not?

$f(x,y)$ is continuous at (x_0,y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

For this problem $(x_0,y_0) = (0,0)$, and $f(x,y) = \begin{cases} \frac{x^2+2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$

so $f(x_0,y_0) = f(0,0) = 2$. On the other hand.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$$

If we take the limit along the line $x=0$, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2} = \lim_{y\to 0} \frac{2y^2}{y^2} = \lim_{y\to 0} 2 = 2.$$

If we take the limit along the line $y=0$, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$$

Therefore $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ does not exist, so it does not equal $f(0,0)=2$,

~~Ther~~ Therefore $f(x,y)$ is not continuous at $(0,0)$.

SID:____________ Name: ____________

2. (6 points) Find an equation of the tangent line to the curve

$$\mathbf{c}(t) = (e^{-t}\cos t, e^{-t}\sin t, e^{-t})$$

at the point $(1, 0, 1)$.

First step: find t such that $c(t) = (1,0,1)$

$$(e^{-t}\cos t, e^{-t}\sin t, e^{-t}) = (1,0,1)$$

$$e^{-t}\cos t = 1, \quad e^{-t}\sin t = 0, \quad \underbrace{e^{-t} = 1}_{t=0}$$

$$c(0) = (e^0\cos(0), e^0\sin(0), e^0) = (1,0,1)$$

Second step: Use formula

$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

with $t_0 = 0$, $c(0) = (1,0,1)$

$$c'(t) = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos(t), -e^{-t})$$

$$c'(0) = (-1-0, 0+1, -1) = (-1, 1, -1)$$

$$l(t) = (1,0,1) + (-1,1,-1)t$$

$$\boxed{l(t) = (1-t, t, 1-t)}$$

SID:____________________ Name: ____________________

3. (7 points) Let $G(s,t) = (u(s,t), v(s,t))$ and $F(u,v)$ be functions, and let $W = F \circ G$ be the composition of F and G, so $W(s,t) = F(u(s,t), v(s,t))$. Suppose that we know that

$$u(1,0) = 2, v(1,0) = 3, \frac{\partial u}{\partial s}(1,0) = -2, \frac{\partial v}{\partial s}(1,0) = 4$$

$$\frac{\partial u}{\partial t}(1,0) = 6, \frac{\partial v}{\partial t}(1,0) = 4, \frac{\partial F}{\partial u}(2,3) = -1, \frac{\partial F}{\partial v}(2,3) = 10$$

What is $DW(1,0)$?

Chain Rule: $DW(1,0) = DF(G(1,0))\, DG(1,0)$

$G(1,0) = (u(1,0), v(1,0)) = (2, 3)$

$$DG(1,0) = \begin{pmatrix} \frac{\partial u}{\partial s}(1,0) & \frac{\partial u}{\partial t}(1,0) \\ \frac{\partial v}{\partial s}(1,0) & \frac{\partial v}{\partial t}(1,0) \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 4 & 4 \end{pmatrix}$$

$$DF(G(1,0)) = DF(2,3) = \left(\frac{\partial F}{\partial u}(2,3) \quad \frac{\partial F}{\partial v}(2,3) \right) = (-1 \quad 10)$$

$$DW(1,0) = DF(G(1,0))\, DG(1,0) = (-1 \quad 10) \begin{pmatrix} -2 & 6 \\ 4 & 4 \end{pmatrix} = (2+40 \quad -6+40) = (42 \quad 34)$$

$$\boxed{DW(1,0) = (42 \quad 34)}$$

SID:________________ Name: ________________

4. Let $f(x,y) = \frac{x^3}{3} + \frac{y^2}{2} - x + 2y + 7$.

   (a) (5 points) Find all the critical points of $f(x,y)$.

$\frac{\partial f}{\partial x} = x^2 - 1, \quad \frac{\partial f}{\partial y} = y + 2$

$0 = x^2 - 1 \qquad y = -2$

$1 = x^2$

$\pm 1 = x$

$\boxed{\text{critical points: } (1,-2), (-1,-2)}$

   (b) (5 points) Determine which critical points are local minima, local maxima, and saddle points of $f(x,y)$.

$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 1, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 0$

point $(1,-2)$: $D = (2\cdot 1)\cdot(1) - 0 = 2 > 0$

$\frac{\partial^2 f}{\partial x^2}(1,-2) = 2$

$\boxed{(1,-2) \text{ is a local minima}}$

point $(-1,-2)$: $D = (2\cdot -1)\cdot(1) - 0 = -2 < 0$

$\boxed{(-1,-2) \text{ is a saddle point}}$

SID:__________________ Name: __________________

5. (8 points) Find the point(s) on the sphere $x^2 + y^2 + z^2 = 1$, where the tangent plane is parallel to the plane $2x + y + z = 7$. Determine the equation of the tangent plane at each of these points.

Normal to tangent plane ~~on~~ to sphere: $(2x, 2y, 2z)$ ~~[illegible]~~

$\left(f(x,y,z) = x^2+y^2+z^2, \quad \nabla f = (2x, 2y, 2z) \right)$

Normal to plane $2x+y+z=7$: $(2,1,1)$

plane with normal $(2x, 2y, 2z)$ is parallel to plane with normal $(2,1,1)$, if

$$(2x, 2y, 2z) = \lambda (2,1,1)$$

for some $\lambda \neq 0$.

$$2x = 2\lambda, \quad 2y = \lambda, \quad 2z = \lambda$$

$$x = \lambda \qquad y = \frac{\lambda}{2}, \qquad z = \frac{\lambda}{2}$$

(x,y,z) needs to be on sphere $x^2+y^2+z^2=1$:

$$\lambda^2 + \frac{\lambda^2}{4} + \frac{\lambda^2}{4} = 1, \qquad \frac{3\lambda^2}{2} = 1, \qquad \lambda^2 = \frac{2}{3}, \quad \lambda = \pm\sqrt{\frac{2}{3}}$$

$$x = \pm\sqrt{\frac{2}{3}}, \quad y = \pm\frac{1}{2}\sqrt{\frac{2}{3}}, \quad z = \pm\frac{1}{2}\sqrt{\frac{2}{3}}$$

Two points are $\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\sqrt{\frac{2}{3}}, \frac{1}{2}\sqrt{\frac{2}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{2}{3}}\right)$

Equation of tangent plane at $\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\sqrt{\frac{2}{3}}, \frac{1}{2}\sqrt{\frac{2}{3}}\right)$: $0 = 2\sqrt{\frac{2}{3}}\left(x - \sqrt{\frac{2}{3}}\right) + \sqrt{\frac{2}{3}}\left(y - \frac{1}{2}\sqrt{\frac{2}{3}}\right) + \sqrt{\frac{2}{3}}\left(z - \frac{1}{2}\sqrt{\frac{2}{3}}\right)$

at $\left(-\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{2}{3}}\right)$: $0 = -2\sqrt{\frac{2}{3}}\left(x + \sqrt{\frac{2}{3}}\right) - \sqrt{\frac{2}{3}}\left(y + \frac{1}{2}\sqrt{\frac{2}{3}}\right) + \sqrt{\frac{2}{3}}\left(z + \frac{1}{2}\sqrt{\frac{2}{3}}\right)$

SID:

Name:

6. Let $f(x, y) = x^2 + y^2 + 3$ and let S be the graph of f.

(a) (4 points) Is the vector $(1, 1, 1)$ a tangent vector to S at the point $(1, -1, 5)$? If so, write down a curve $\mathbf{c}(t)$ on S such that $\mathbf{c}(0) = (1, -1, 5)$ and $\mathbf{c}'(0) = (1, 1, 1)$. If not, explain why not.

Equation of tangent plane to S at $(1,-1,5)$: $z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

with $(x_0,y_0) = (1,-1)$, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$

$z = 5 + 2(x-1) - 2(y+1)$, $0 = -z + 5 + 2x - 2 - 2y - 2$, $0 = 2x - 2y - z + 1$

normal to tangent plane: $(2,-2,-1)$. $(1,1,1)$ is a tangent vector if $(1,1,1)$ is orthogonal to the normal vector.

$(2,-2,-1)\cdot(1,1,1) = 2 - 2 - 1 = -1$ so $(1,1,1)$ is <u>not</u> a tangent vector. since $(1,1,1)$ is not orthogonal to $(2,-2,-1)$.

(b) (4 points) Is the vector $(1, 1, 0)$ a tangent vector to S at the point$(1, -1, 5)$? If so, write down a curve $\mathbf{c}(t)$ on S such that $\mathbf{c}(0) = (1, -1, 5)$ and $\mathbf{c}'(0) = (1, 1, 0)$. If not, explain why not.

Same reasoning: $(1,1,0)$ is a tangent vector if $(1,1,0)$ is orthogonal to $(2,-2,-1)$. $(1,1,0)\cdot(2,-2,-1) = 2-2+0 = 2$ so $(1,1,0)$ <u>is</u> a tangent vector.

$c(t) = (x_0+tv_1, y_0+tv_2, f(x_0+tv_1, y_0+tv_2))$ is a curve on S such that

$c(0) = (x_0, y_0, f(x_0,y_0))$ and

$c'(0) = (v_1, v_2, \nabla f(x_0,y_0)\cdot(v_1,v_2))$

$(x_0,y_0) = (1,-1)$, $(v_1,v_2) = (1,1)$, then

$c(t) = (1+t, -1+t, f(1+t,-1+t)) = (1+t, -1+t, (1+t)^2 + (-1+t)^2 + 3)$

$= (1+t, -1+t, 2t^2+5$

$\boxed{c(t) = (1+t, -1+t, 2t^2+5) \text{ is on } S \text{ and satisfies } c(0) = (1,-1,5) \text{ and } c'(0) = (1,1,0).}$