Midterm

- Monday during lecture
- NO CALCULATORS or electronic aids
- One 8.5×11 in. hand-written page of notes front and back

7 questions:
- 4 integrals
  - 2 regular
  - 2 change of variables
- 2 short questions from 6.1 and 6.2
- 1 extra credit question

50 total points plus possible 5 extra credit points

4.3 Vector Fields

A **vector field** in $\mathbb{R}^2$ is a function

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$F(x,y) = (F_1(x,y),\ F_2(x,y))$$

A **vector field** in $\mathbb{R}^3$ is a function

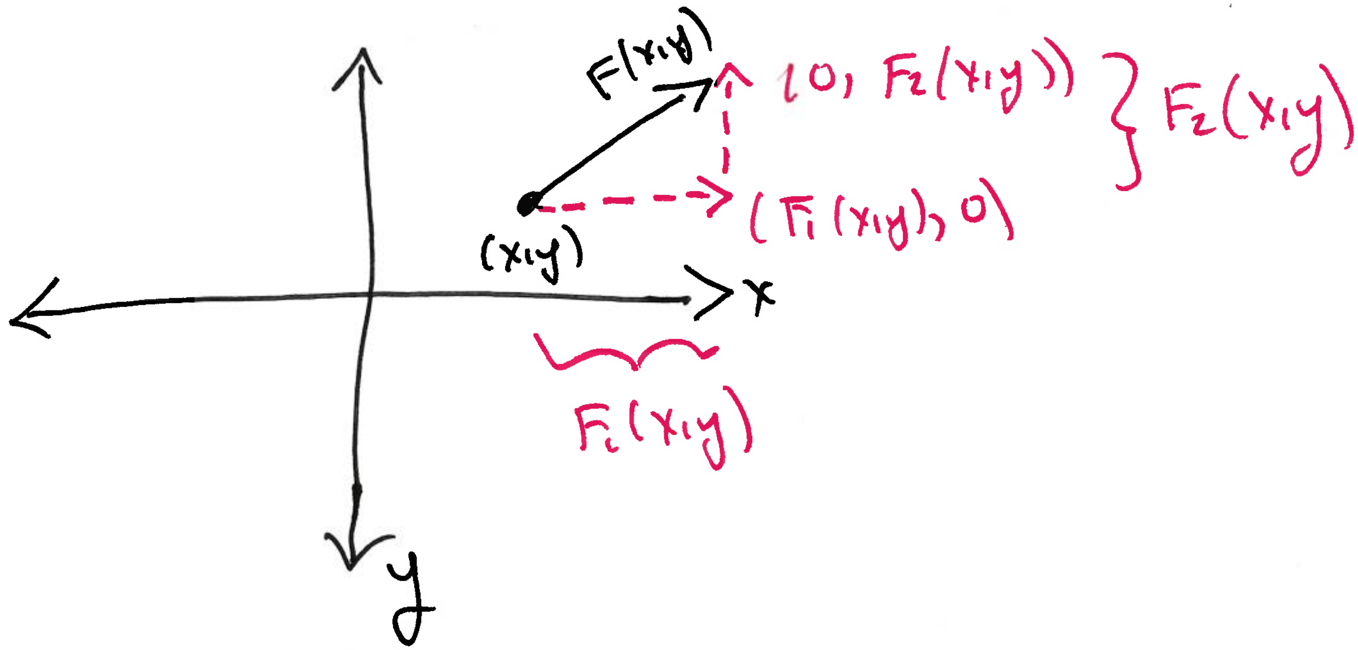
$$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$F(x,y,z) = (F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z))$$

(*) Vector fields model forces and motion in space. (*)

Vector fields attach to each point (x,y,z) a vector $F(x,y,z) = (F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z))$.

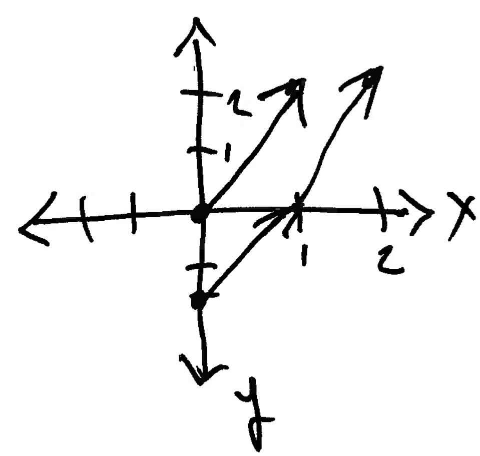
Drawing a vector field $F(x,y) = (F_1(x,y), F_2(x,y))$



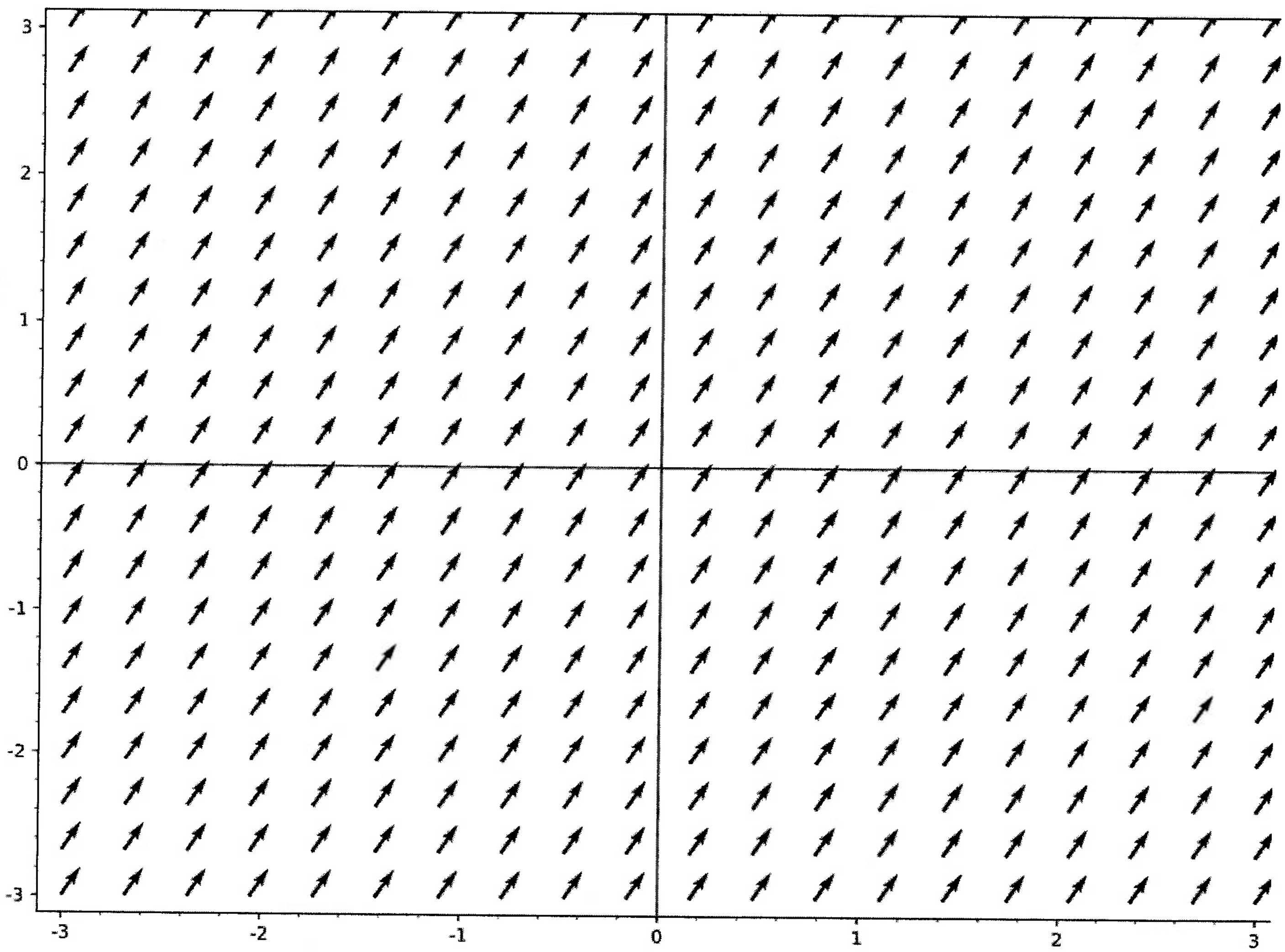
Examples

1. Constant vector field

$$F(x,y) = (1, 2)$$



```
x,y = var('x y')
plot_vector_field((1,2), (x,-3,3), (y,-3,3), figsize = 10)
```



$F(x,y) = (1,2)$

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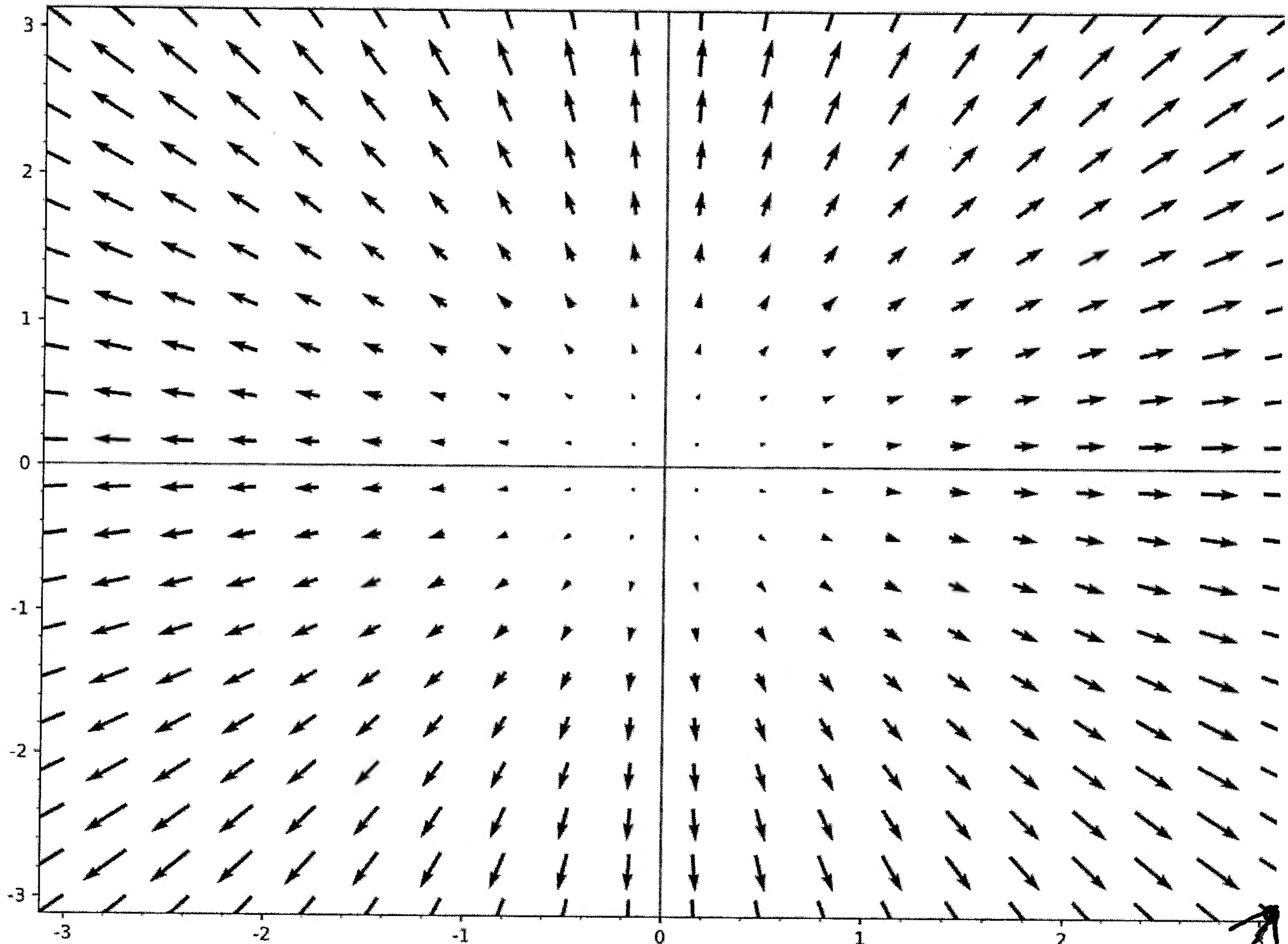
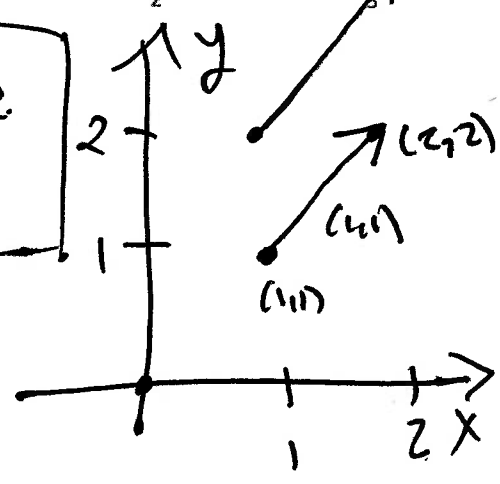
```
x,y = var('x y')
plot_vector_field((x,y), (x,-3,3), (y,-3,3), figsize = 10)
```



sagemath program plotted these
cocalc.com

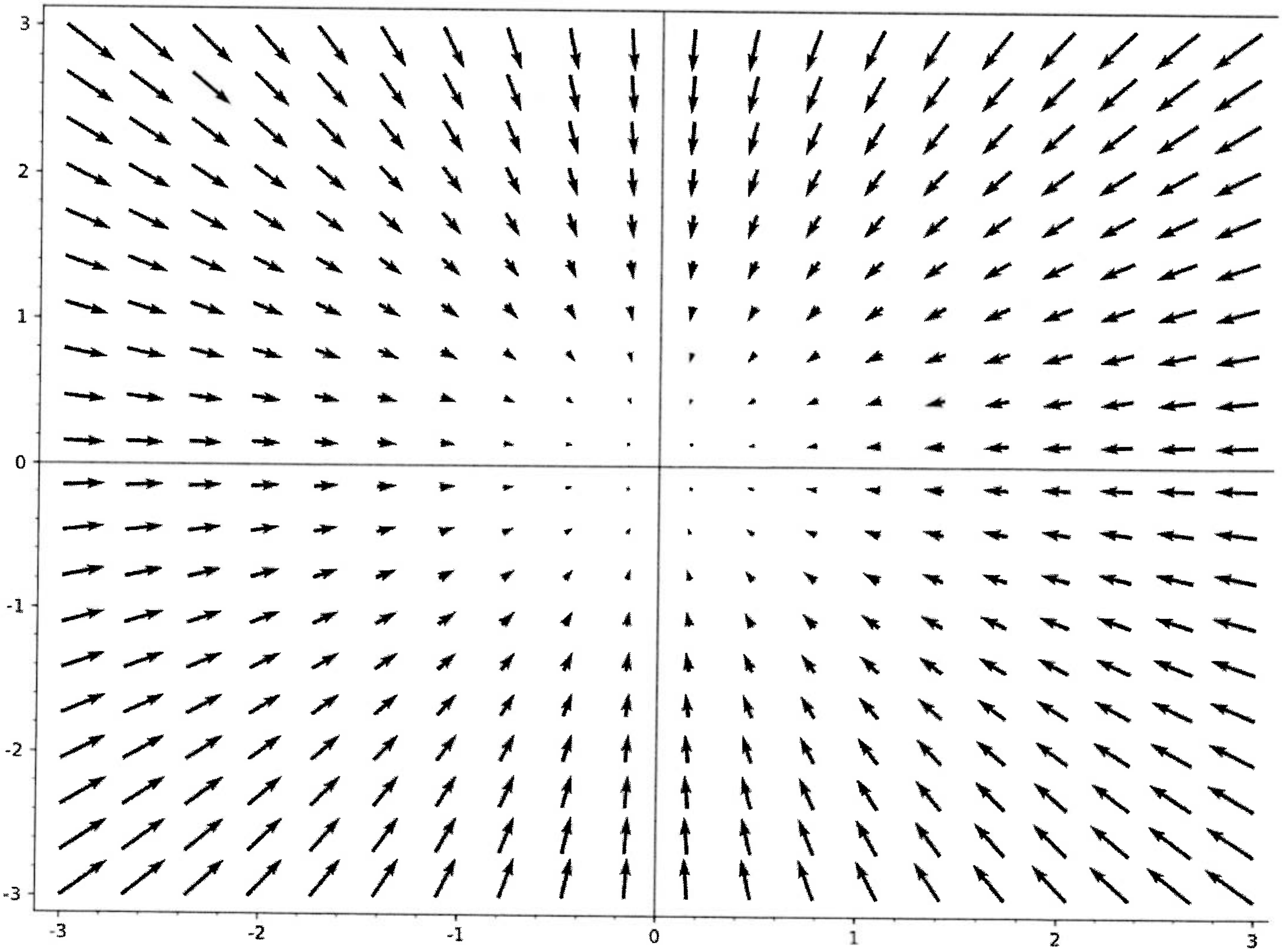


$F(x,y) = (x,y)$

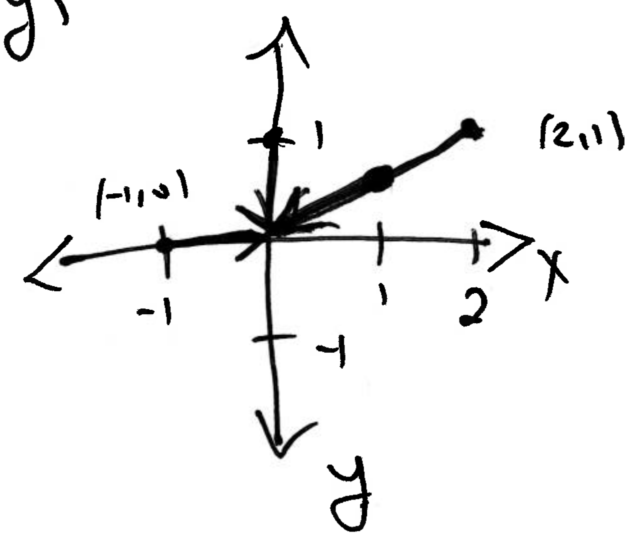
$F(1,1) = (1,1)$

$F(1,2) = (1,2)$

```
x,y = var('x y')
plot_vector_field((-x, -y), (x,-3,3), (y,-3,3), figsize = 10)
```



$F(x,y) = (-x,-y)$



$F(-1,0) = (1,0)$

$F(2,1) = (-2,-1)$

$F(1,1) = (-1,-1)$

$F(0,1) = (0,-1)$

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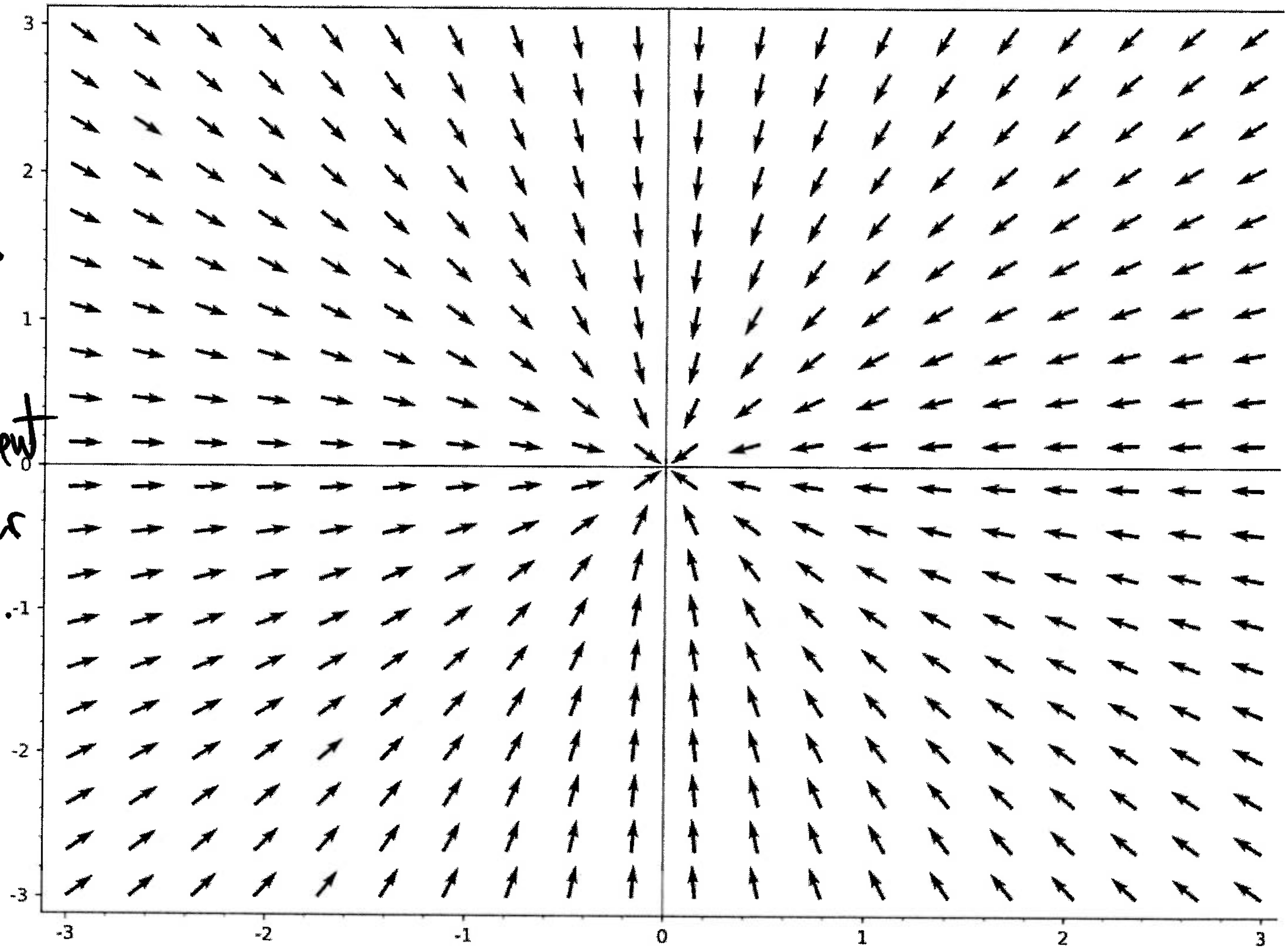
```
x,y = var('x y')
plot_vector_field((-x/sqrt(x^2 + y^2), -y/sqrt(x^2 + y^2)), (x,-3,3), (y,-3,3), figsize = 10)
```

This is a gradient vector field.



$$F(x,y) = \frac{(-x,-y)}{\|(x,y)\|} = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right) = (F_1(x,y), F_2(x,y))$$

$$\|(x,y)\| = \sqrt{x^2+y^2}$$

Gravity (??): $F(x,y) = \dfrac{(-x,-y)}{\|(x,y)\|^{3/2}}$ if sun is at origin

arrows get shorter the further from the origin.

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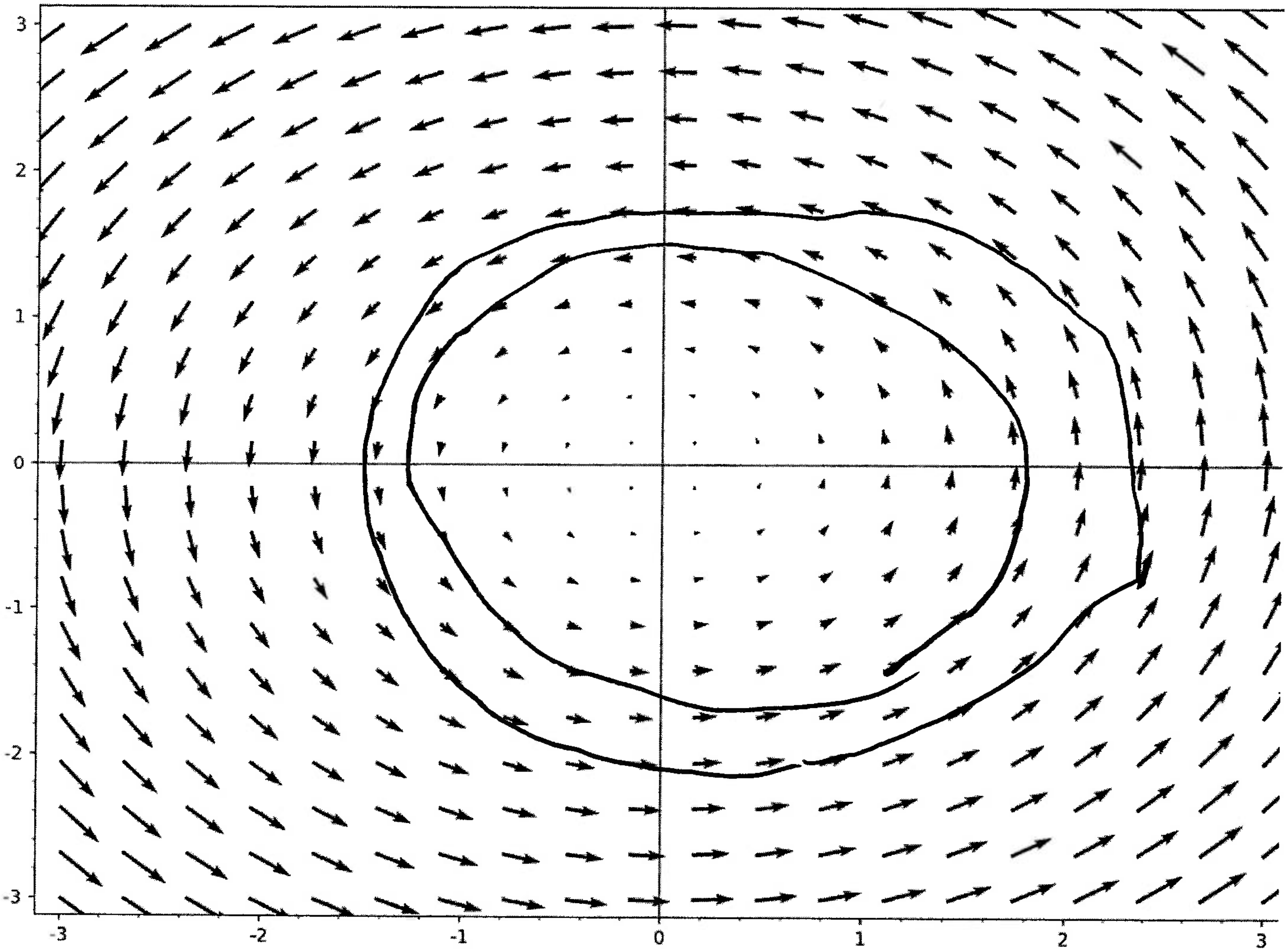
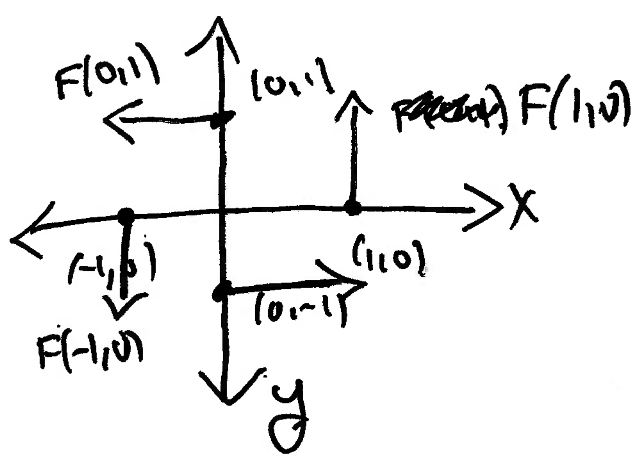
```
x,y = var('x y')
plot_vector_field((-y, x), (x,-3,3), (y,-3,3), figsize = 10)
```



$F(x,y) = (-y, x)$

$F(1,0) = (0, 1)$
$F(0,1) = (-1, 0)$
$F(-1,0) = (0, -1)$
$F(0,-1) = (1, 0)$



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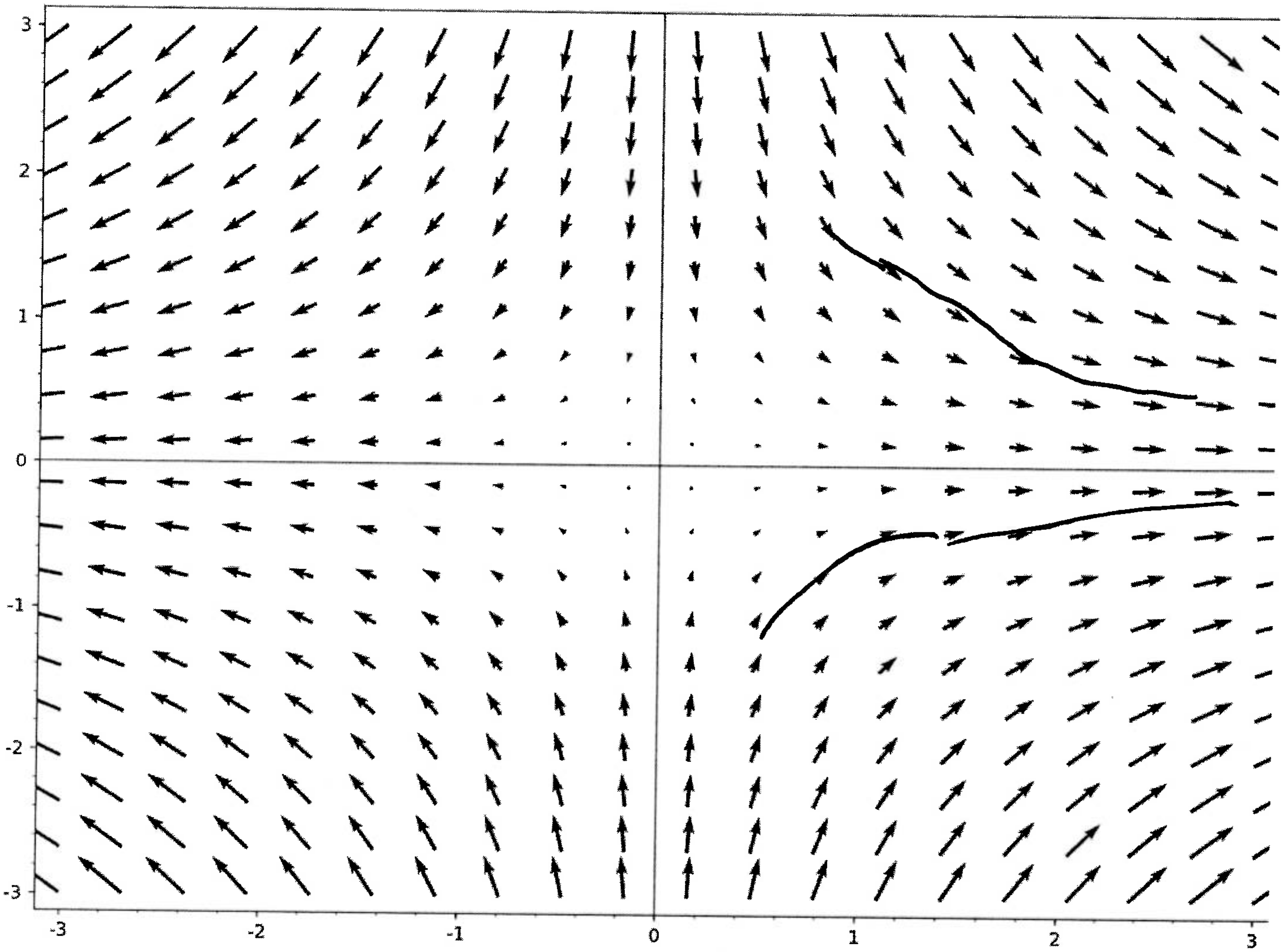
```
x,y = var('x y')
plot_vector_field((x, -y), (x,-3,3), (y,-3,3), figsize=10)
```



$F(x,y) = (x, -y)$

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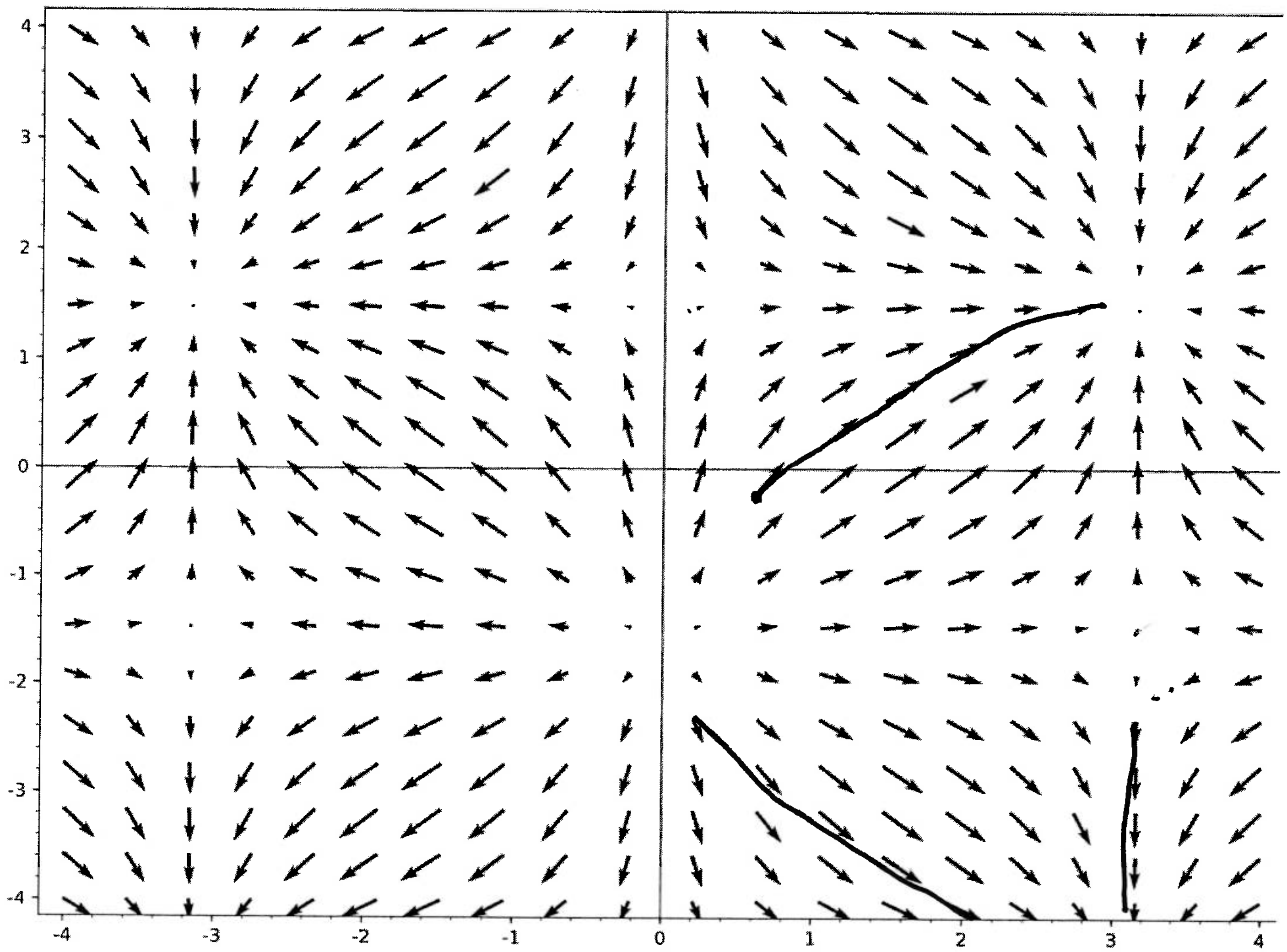
```
x,y = var('x y')
plot_vector_field((sin(x),cos(y)), (x,-4,4), (y,-4,4), figsize = 10)
```



$F(x,y) = (\sin(x), \cos(y))$

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Gradient Vector Field

$f(x,y,z)$ (or $f(x,y)$) scalar valued function

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right)$$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$$

Gradient vector field of f.

Example

$$f(x,y,z) = -\sqrt{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{-x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{-y}{\sqrt{x^2+y^2}}$$

$$\nabla f = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right)$$

$F(x,y)$ is a gradient vector field if there exists a C^2 function $f(x,y)$ such that $\nabla f = F$.

Non-example

Is $F(x,y) = (-y, x)$ a gradient vector field?

No, and here is a proof why not.

Assume $F(x,y) = (-y,x)$ is a gradient vector field.

Then there exists $f(x,y)$ such that $\nabla f = F$, i.e.

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-y, x)$$

So $\frac{\partial f}{\partial x} = -y$ and $\frac{\partial f}{\partial y} = x$. Then

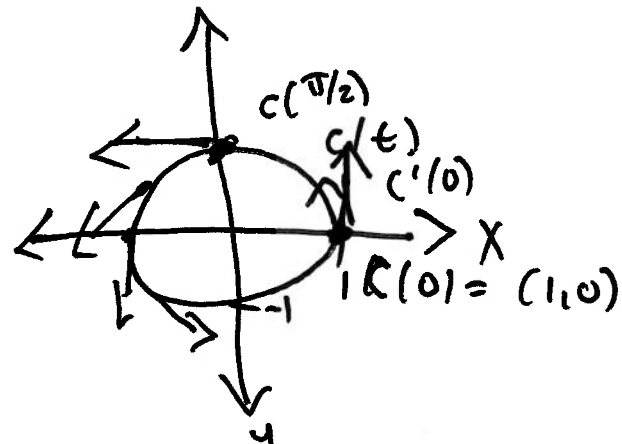
$$\frac{\partial^2 f}{\partial y \partial x} = -1 \qquad \frac{\partial^2 f}{\partial x \partial y} = 1$$

but $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. This is a contradiction

so no f can exist.

$c(t) = (\cos t, \sin t)$
$0 \leq t \leq 2\pi$

$c(t)$ parametrizes the unit circle



$c'(t) = (-\sin t, \cos t)$ tangent vectors to $c(t)$
$c'(0) = (0,1)$
$c'(\frac{\pi}{2}) = (-1, 0)$

Remember $F(x,y) = (-y, x)$

Relation: $\underset{(-\sin t, \cos t)}{\overset{\parallel}{c'(t)}} = F(c(t)) = F(\cos t, \sin t) = (-\sin t, \cos t)$

The vector field $F(x,y) = (-y, x)$ is tangent to the curve $c(t)$.

The curve $c(t)$ flows along the vector field $F(x,y)$.

<u>Def</u>: If F is a vector field, a <u>flow line</u> for F is a curve $c(t)$ such that

$$c'(t) = F(c(t)).$$