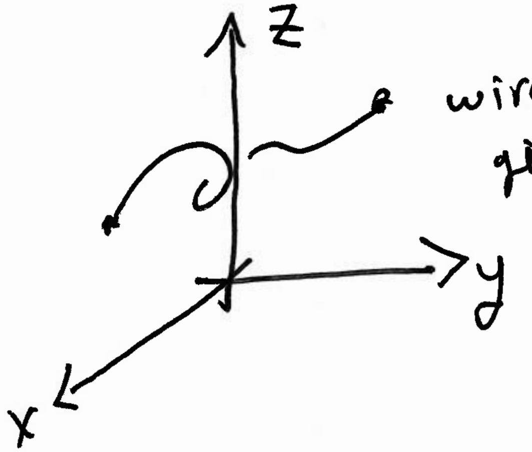
7.1 The Path Integral

Motivating Example



wire in space given by path

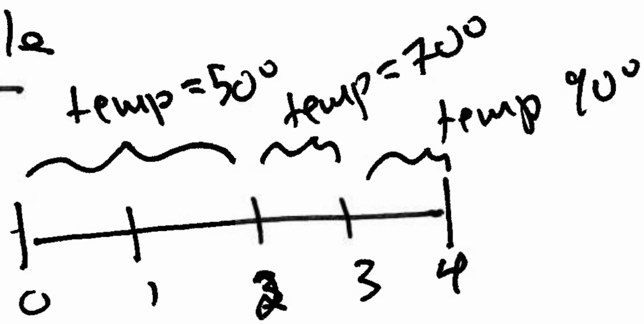
$c(t) = (x(t), y(t), z(t)), \quad a \le t \le b$

$f(x,y,z)$ = temperature at point (x,y,z)

$f(c(t))$ = temperature on wire

Question: what's the average temperature on the wire?

Example



average temp $65°$

$$= \frac{50\cdot 2 + 70\cdot 1 + 90\cdot 1}{4}$$

$c(t) = (x(t), y(t), z(t)), \quad a \le t \le b$

Def: The path integral or the integral of $f(x,y,z)$ along the the path c is

$$\int_c f\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

$$= \int_a^b \underbrace{f(c(t))}_{\text{Value of } f \text{ at } c(t)} \underbrace{\|c'(t)\|\,dt}_{\text{length of infinitesimal peice (ie?) of the curve at } c(t)}$$

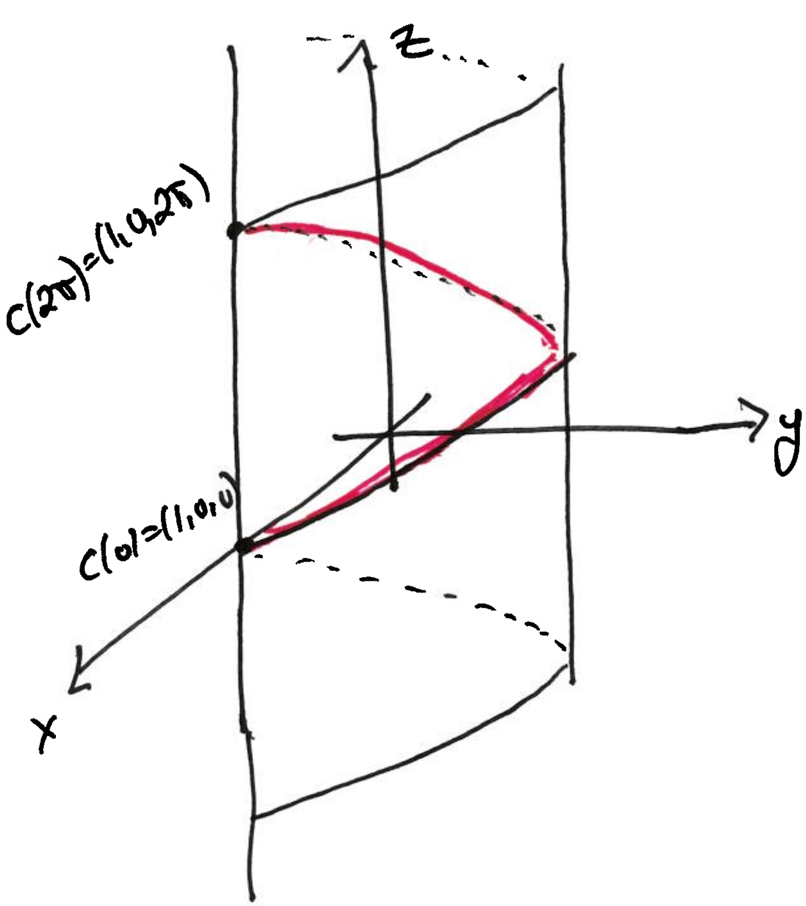
Fine print: $c(t)$ must be C^1 and $f(c(t))$ must be continuous for the integral to be defined.

Example

$$c(t) = (\cos t, \sin t, t), \quad 0 \le t \le 2\pi$$

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{Evaluate } \int_c f \, ds.$$

$c(t)$ is a helix



$$\int_0^{2\pi} f(c(t)) \, \|c'(t)\| \, dt$$

$$c'(t) = (-\sin t, \cos t, 1)$$

$$\|c'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1^2}$$

$$= \sqrt{2}$$

$$f(c(t)) = f(\cos t, \sin t, t)$$

$$= \cos^2 t + \sin^2 t + t^2$$

$$= 1 + t^2$$

$$\int_c f\,ds = \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt$$

$$= \sqrt{2}\left(t + \frac{t^3}{3} \Big|_0^{2\pi} \right)$$

$$= \boxed{\sqrt{2}\cdot 2\pi + \frac{\sqrt{2}\,(2\pi)^3}{3}}$$

<u>Average Value</u> of $f(x,y,z)$ over $c(t) = (x(t), y(t), z(t))$, $a \le t \le b$

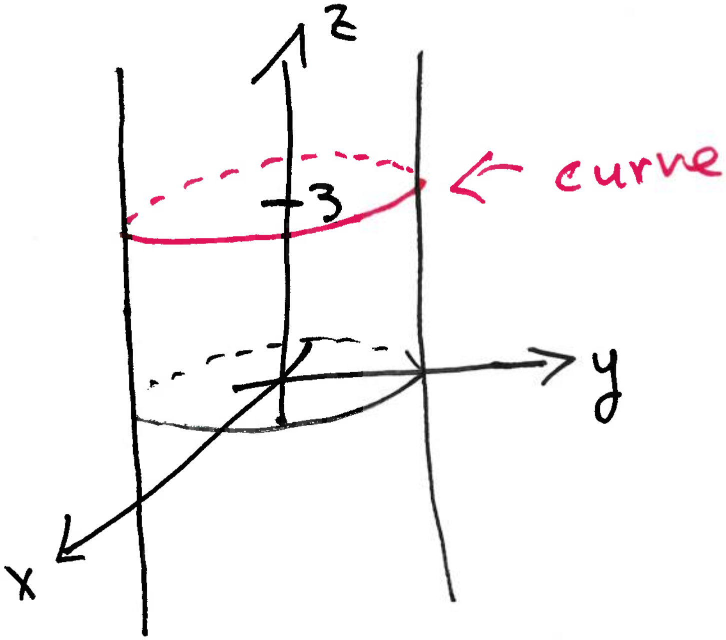
is

$$\frac{\int_c f\,ds}{\text{length of } c(t)} = \frac{\int_a^b f(c(t))\,\|c'(t)\|\,dt}{\int_a^b \|c'(t)\|\,dt}$$

Parametrizing Curves

Example

Parametrize the intersection of $z=3$ (plane) with $\frac{x^2}{9}+\frac{y^2}{16}=1$ (elliptic cylinder)



What $c(t)=(x(t),y(t),z(t))$ ~~gives~~ parametrizes the curve?

$x=3\cos t$, $y=4\sin t$, $0\le t\le 2\pi$

$$\frac{(3\cos t)^2}{9}+\frac{(4\sin t)^2}{16}$$
$$=\frac{9\cos^2 t}{9}+\frac{16\sin^2 t}{16}$$
$$=1$$

Answer: $c(t)=(3\cos t, 4\sin t, 3)$
$0\le t\le 2\pi$

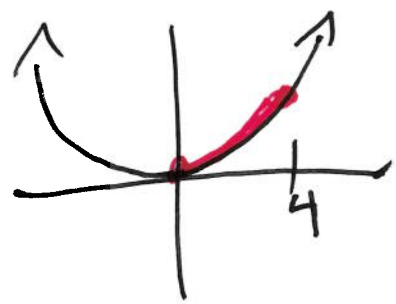
Parametrizing the graph of a function in $\mathbb{R}^2$

Parametrize $y = f(x)$, $a \le x \le b$ in $\mathbb{R}^2$

what $c(t) = (x(t), y(t))$ parametrizes the graph of f?

$$c(t) = (t, f(t)), \quad a \le t \le b$$

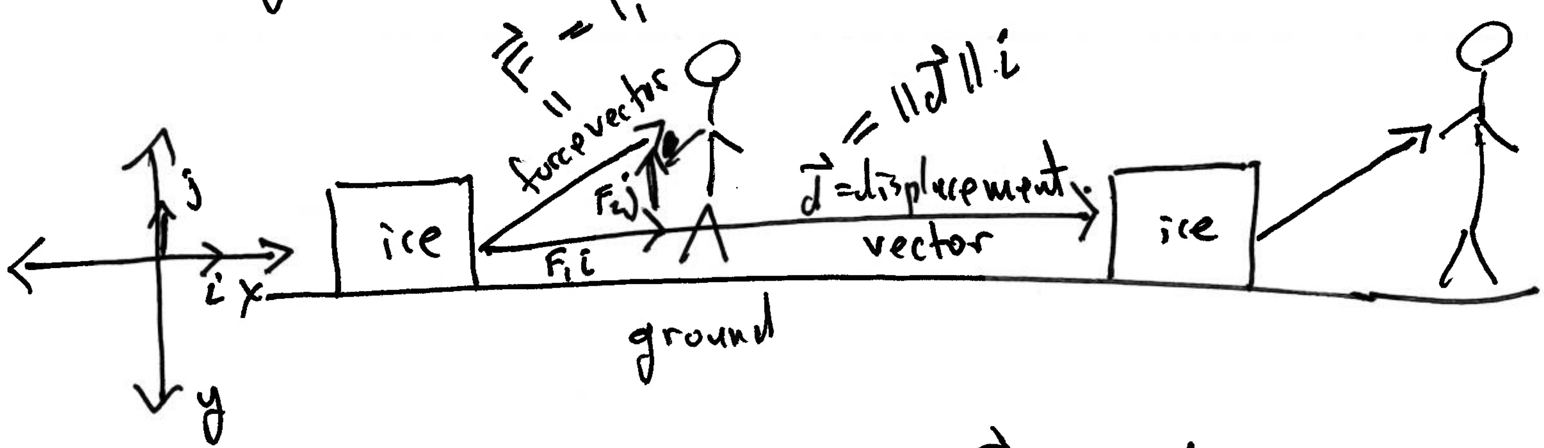
Example $y = x^2$, $0 \le x \le 4$



$$c(t) = (t, t^2), \quad 0 \le t \le 4$$

7.2 Line Integrals

Palling Ice $\qquad\qquad i = (1,0), \quad j = (0,1)$



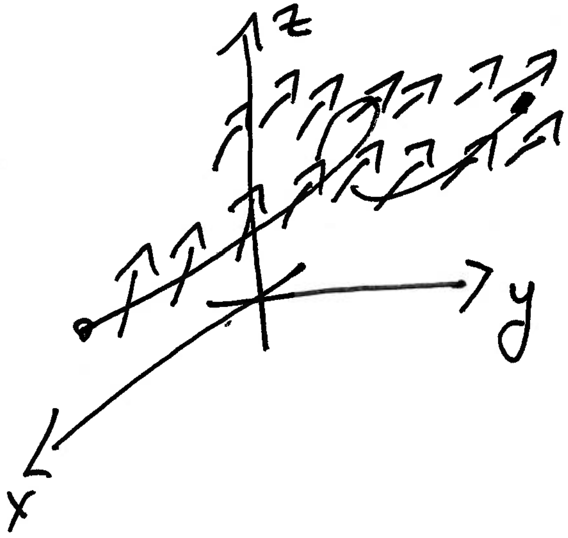
How much work does $\vec{F}$ do on the ice?

$$\vec{F} \cdot \vec{d} = (F_1 i + F_2 j) \cdot \|\vec{d}\| i = F_1 \|\vec{d}\|$$

In general: $c(t) = (x(t), y(t), z(t)) \quad a \le t \le b$ path

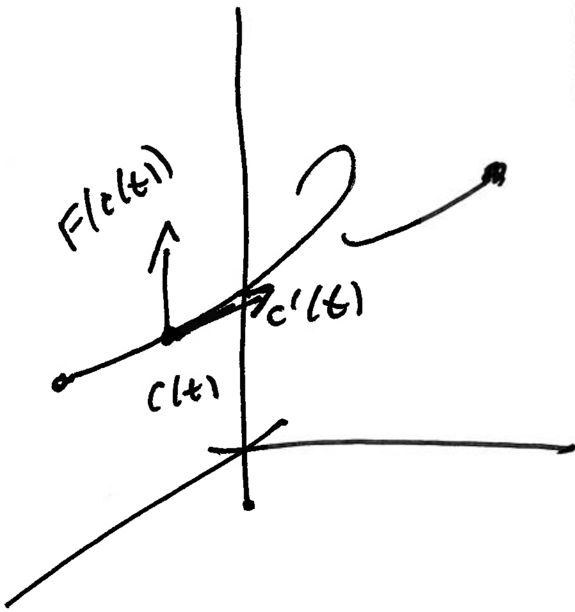
$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ vector field

What is the work done by F on c?



Def: $F(x,y,z)$ – vector field, $c(t)$, $a \le t \le b$ curve. The line integral of F along c is

$$\int_c F \cdot d\vec{s} = \int_a^b \underbrace{F(c(t))}_{\substack{\text{force on} \\ c(t) \text{ at} \\ \text{time } t}} \cdot \underbrace{c'(t)\,dt}_{\substack{\text{infinitesimal direction} \\ \text{or displacement of} \\ c(t) \text{ at time } t}}$$



Fine print: F is continuous on $c(t)$ and $c(t)$ is C^1.