7.2 Line Integrals

F - vector field, $c(t)$, $a \le t \le b$ curve

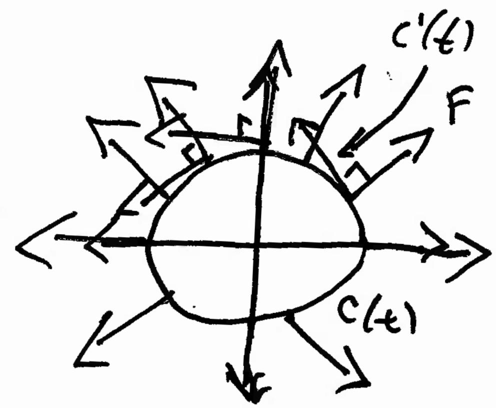
$$\int_c F \cdot d\vec{s} = \int_a^b F(c(t)) \cdot c'(t)\, dt \qquad \text{work of } F \text{ on } c$$

Examples

① $F(x,y) = (x,y)$, $\quad c(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

Calculate

$$\int_c F \cdot d\vec{s}$$



$$F(c(t)) = F(\cos t, \sin t) = (\cos t, \sin t)$$

$$c'(t) = (-\sin t, \cos t)$$

$$\int_c F \cdot d\vec{s} = \int_0^{2\pi} (\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\cos t \sin t + \sin t \cos t \, dt = \int_0^{2\pi} 0\, dt = 0$$

<u>Notation</u>: $F = (F_1, F_2, F_3)$

$$\int_c F \cdot d\vec{s} = \int_c F_1 dx + F_2 dy + F_3 dz$$

$d\vec{s} = (dx, dy, dz)$

(2) Evaluate

$$\int_c x^2 dx + xy\,dy + dz$$

where $c(t) = (t, t^2, 1)$, $0 \le t \le 1$.

$F(x,y,z) = (x^2, xy, 1)$

$$\int_C x^2\,dx + xy\,dy + dz = \int_0^1 t^2\cdot 1 + t\cdot t^2\cdot 2t + 0\,dt$$

using $x=t,\ y=t^2,\ z=1$ in definition $c(t)=(t,t^2,1)$

$$= \int_0^1 t^2 + 2t^4\,dt$$

$$= \frac{t^3}{3} + \frac{2t^5}{5}\Big|_0^1$$

$$= \boxed{\frac{1}{3} + \frac{2}{5}}$$

(3) $\int_C \cos z \, dx + e^x dy + e^y dz$ where

$c(t) = (1, t, e^t), \quad 0 \le t \le 2.$

<u>Solution</u>: $x = 1, \quad y = t, \quad z = e^t$

$dx = 0 \cdot dt, \quad dy = 1 \cdot dt, \quad dz = e^t dt$

$$\int_0^2 \cos(e^t) \cdot 0 + e^1 \cdot 1 + e^t e^t \, dt =$$

$$= \int_0^2 e + e^{2t} \, dt = et + \frac{1}{2} e^{2t} \Big|_0^2$$

$$= \boxed{2e + \frac{e^4}{2} - 0 - \frac{1}{2}}$$

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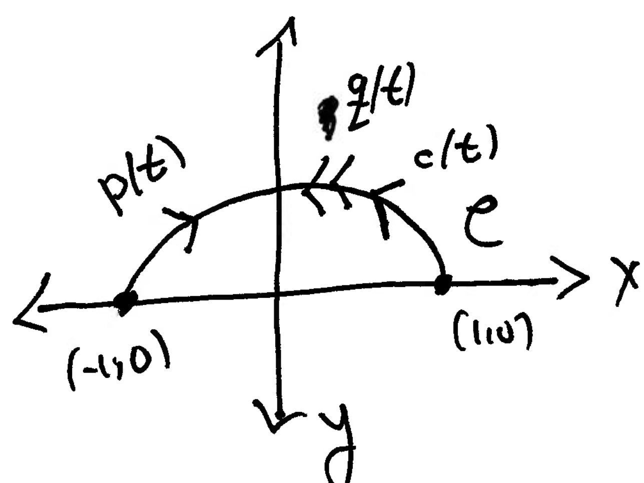
Reparametrizations

There is more than one way to parametrize a curve.

Do path or line integrals depend on the parametrization of the curve?

Examples

$\mathcal{C}$ = curve given by equations $x^2+y^2=1$ and $y \geq 0$.



$c(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$

$p(t) = (\cos(\pi - t), \sin(\pi - t))$, $0 \leq t \leq \pi$

$q(t) = (\cos(2t), \sin(2t))$, $0 \leq t \leq \frac{\pi}{2}$

Path Integral for arc length of e $(f(x,y)=1)$

$$\int_0^{\pi} \|c'(t)\|\,dt = \int_0^{\pi} \sqrt{\sin^2 t + \cos^2 t}\;dt = \int_0^{\pi} 1\,dt = \pi$$

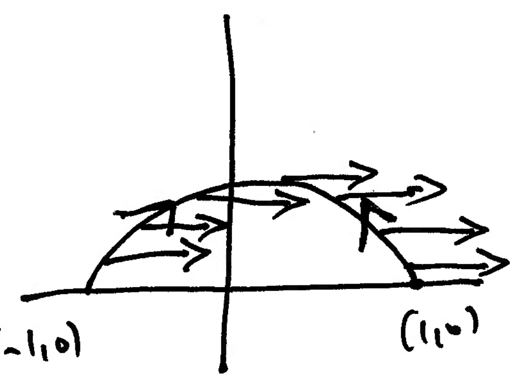
$$\int_0^{\pi} \|p'(t)\|\,dt = \int_0^{\pi} \sqrt{\sin^2(\pi-t) + \cos^2(\pi-t)}\;dt = \pi$$

$$\int_0^{\pi/2} \|q'(t)\|\,dt = \int_0^{\pi/2} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}\;dt = 2\int_0^{\pi/2} dt = 2\cdot\frac{\pi}{2} = \pi$$

All give same answer.

<u>Line Integral</u> $\quad F(x,y) = (1,0)$ constant vector field.

$$\int_C F\cdot d\vec{s} = \int_0^{\pi} F(c(t))\cdot c'(t)\,dt = \int_0^{\pi} (1,0)\cdot(-\sin t, \cos t)\,dt$$



$$\boxed{\cancel{\int_p F\cdot d\vec{s} =}} = \int_0^{\pi} -\sin t + 0\,dt =$$

$$= \cos t \Big|_0^{\pi} = -1-1 = \boxed{-2}$$

$$\int_P F \cdot d\vec{s} = \int_0^{\pi} (1,0) \cdot (\sin(\pi - t), -\cos(\pi - t))\, dt$$

$$p(t) = (\cos(\pi - t), \sin(\pi - t))$$

$$p'(t) = (-\sin(\pi - t) \cdot -1, \cos(\pi - t) \cdot -1)$$

$$= \int_0^{\pi} \sin(\pi - t)\, dt = +\cos(\pi - t)\Big|_0^{\pi} = 1 - -1 = \boxed{2}$$

Let $c(t)$, $p(t)$ be two parametrizations of the same curve $\mathcal{C}$. Let F be a vector field.

Theorem (change of parametrization for line integrals) If $c(t)$ and $p(t)$ go in opposite directions then

$$\int_c F \cdot d\vec{s} = - \int_p F \cdot d\vec{s}.$$

If $c(t)$ and $p(t)$ go in the same direction then

$$\int_c F \cdot d\vec{s} = \int_p F \cdot d\vec{s}.$$

Theorem (change of parametrization for path integrals) Let f be a scalar valued function. Then

$$\int_c f\,ds = \int_p f\,ds.$$