Line Integrals of Gradient Vector Fields

Thm: f - real valued function, $c(t)$, $a \le t \le b$. Then

$$\int_c \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$$

proof sketch: This is the FTOC for the function

$$g(t) = f(c(t)).$$

By chain rule

$$g'(t) = \nabla f(c(t)) \cdot c'(t).$$

Then

$$\int_c \nabla f \cdot d\vec{s} = \int_a^b \nabla f(c(t)) \cdot c'(t)\,dt \overset{\text{FTOC}}{=} g(b) - g(a)$$

$$= f(c(b)) - f(c(a)). \square$$

Example $c(t) = (t^4/4, \sin(t\pi/2), t), \; 0 \le t \le 1$

Evaluate $\int_c y\,dx + x\,dy + 0 \cdot dz.$

$c(0) = (0,0,0)$
$c(1) = (\frac{1}{4}, 1, 1)$

Solution: $F(x,y,z) = (y, x, 0)$

Find f such that $\nabla f = F$.

$$\frac{\partial f}{\partial x} = y \rightsquigarrow f(x,y,z) = xy + f_1(y,z)$$

$$\frac{\partial f}{\partial y} = x \rightsquigarrow f(x,y,z) = xy + f_2(x,z)$$

$$\frac{\partial f}{\partial z} = 0 \rightsquigarrow f(x,y,z) = f_3(x,y)$$

If $f(x,y,z) = xy$, then $\nabla f = (y, x, 0) = F$

Then

$$\int_c y\,dx + x\,dy + 0\cdot dz = f(c(1)) - f(c(0))$$
$$= f(\tfrac{1}{4},1,1) - f(0,0,0)$$
$$= \boxed{\tfrac{1}{4}} \leftarrow = \tfrac{1}{4}\cdot 1 - 0\cdot 0$$

Given vector field $F(x,y,z)$, how do you tell if $f(x,y,z)$ exists such that $\nabla f = F$?

~~Say~~

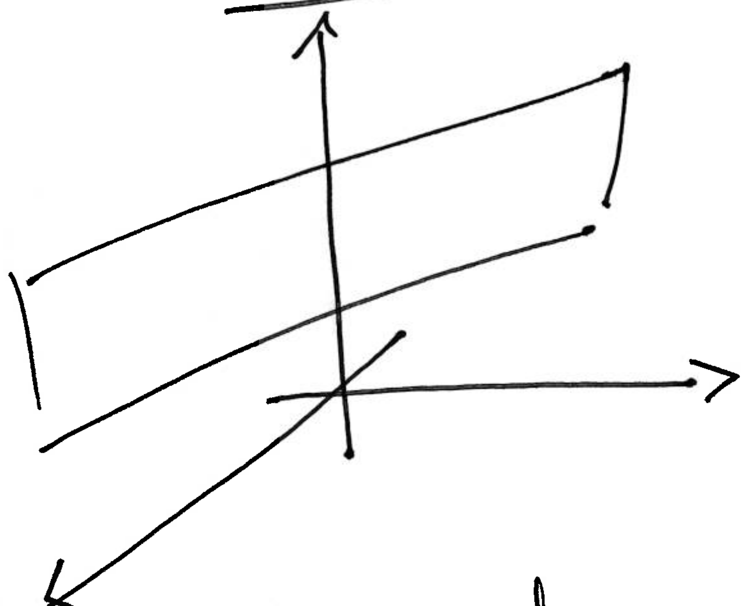
We'll say more about this in section 8.3.

7.3 Parametrized Surfaces

A surface is a 2-dimensional object in $\mathbb{R}^3$.

Examples

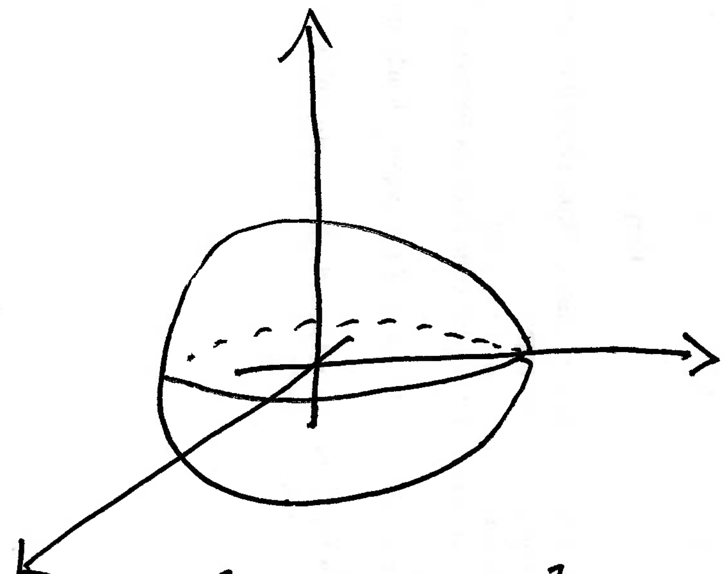
Planes



$ax+by+cz=d$

a,b,c,d constants

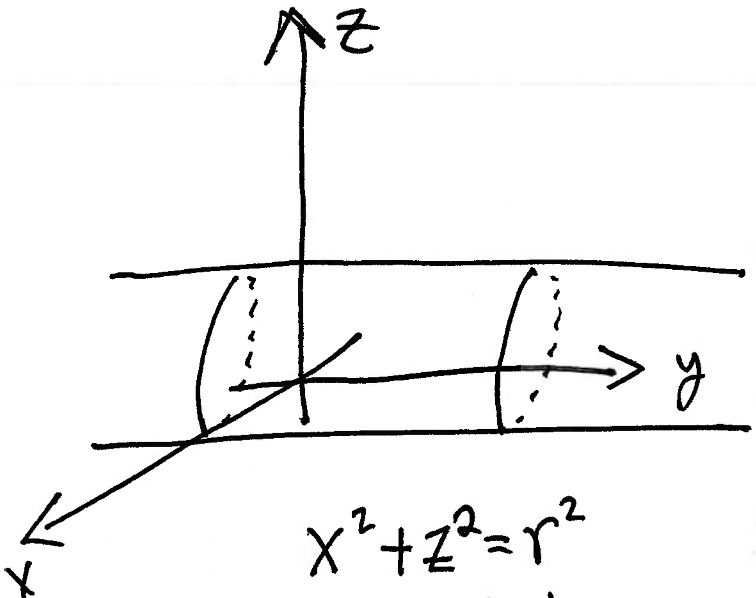
Spheres



$x^2+y^2+z^2=\rho^2$

ρ - constant

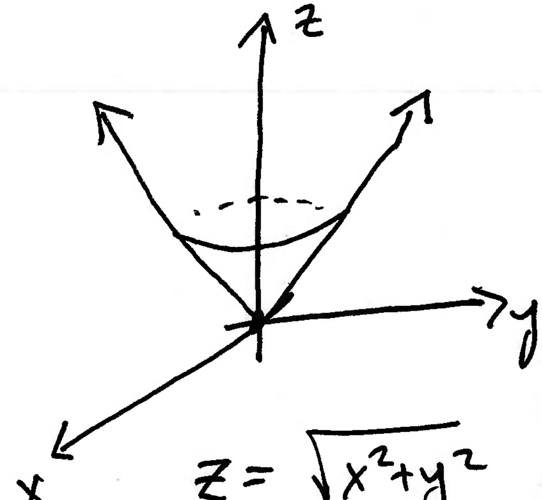
Cylinders



$x^2+z^2=r^2$

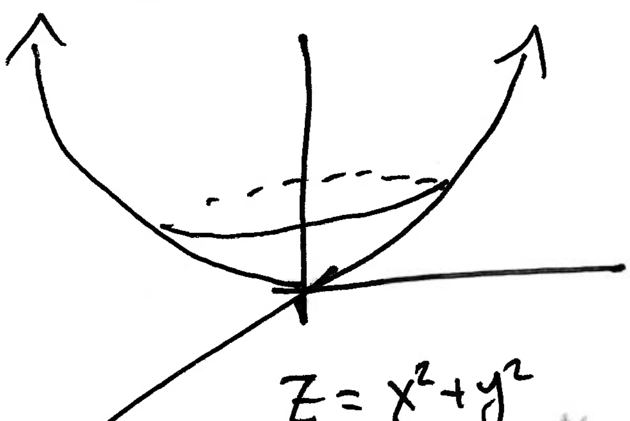
r - constant

Cone



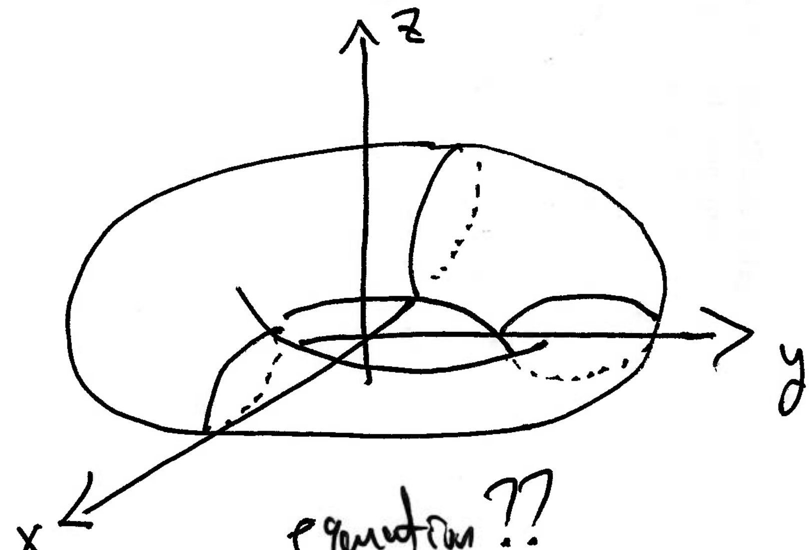
$z=\sqrt{x^2+y^2}$

Paraboloid



$z=x^2+y^2$

Torus (surface of a donut.)



equation??

plane, paraboloid, cone are graphs of functions

sphere, cylinder, torus are not graphs of functions
(they're level sets)

Def: A parametrized surface is a function

$\Phi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$. The surface S corresponding to Φ is its image $S = \Phi(D)$. We can write

$$\Phi(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Fine print: If Φ is differentiable or C^1, we call S a differentiable or C^1 surface.

Examples

(1) Parametrize $ax + by + cz = d$. (assume $c \neq 0$)

Solve for z: $$z = \frac{d - ax - by}{c}$$

Then

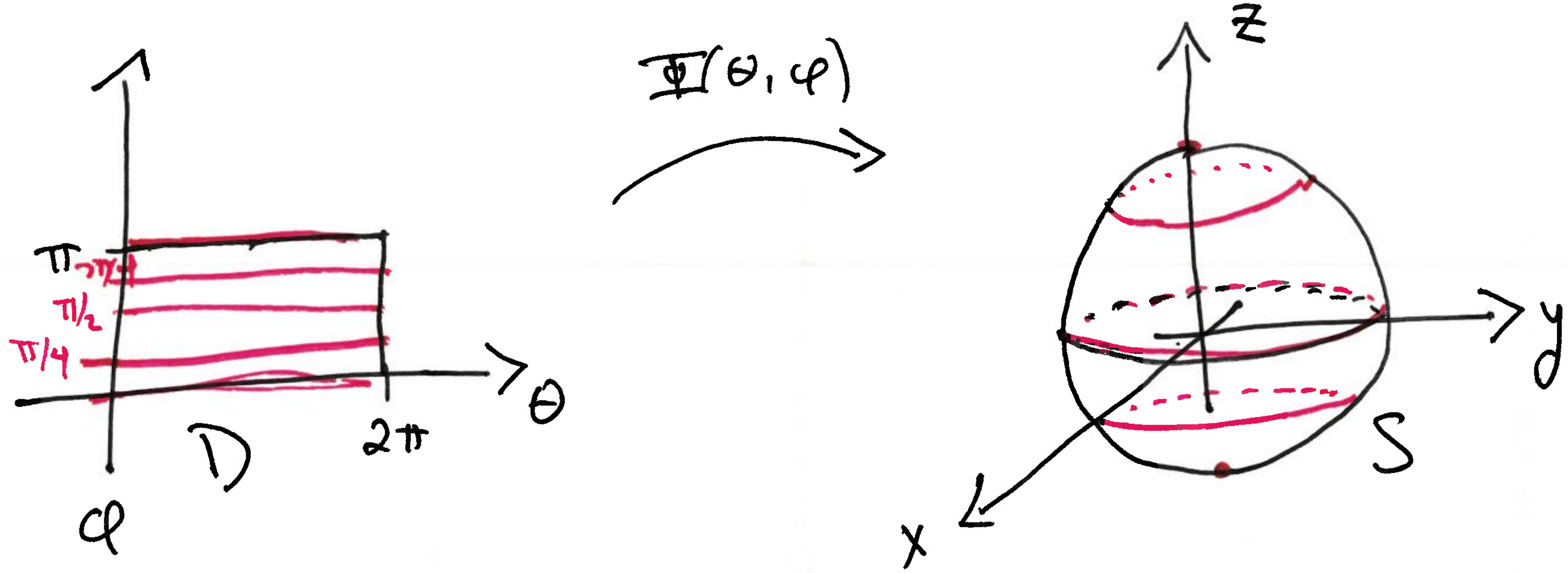
$$\Phi(u,v) = \left(u, v, \frac{d - au - bv}{c}\right), \quad D = \mathbb{R}^2$$

(2) Parametrize $x^2 + y^2 + z^2 = 9$

Use spherical coordinates, $\theta, \varphi, \rho = 3$

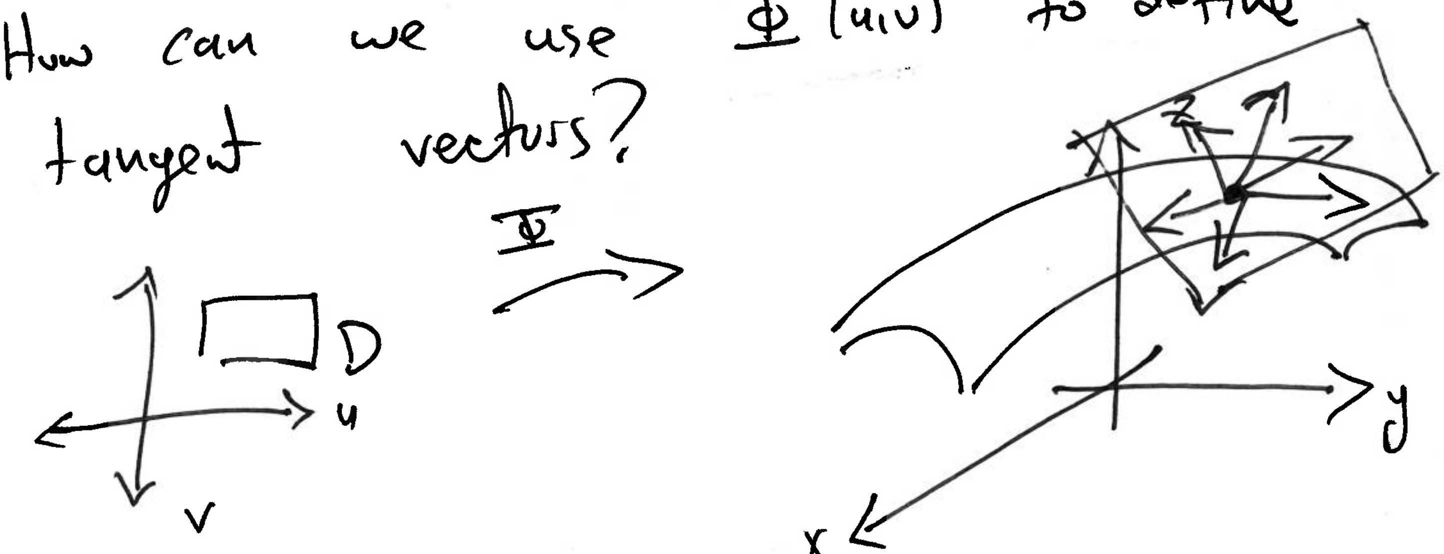
$$\Phi(\theta, \varphi) = (3\sin\varphi\cos\theta, 3\sin\varphi, \sin\theta, 3\cos\varphi)$$

$$D: \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}$$



Tangent Vectors to Parametrized Surfaces.

Question: What is a tangent vector to a surface?
How can we use $\Phi(u,v)$ to define tangent vectors?



$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, (u,v) in D

parametrization of a surface

$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$ tangent vector in u-direction

$T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$ tangent vector in v-direction