7.3 Parametrized Surfaces

$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, (u,v) in $D \subseteq \mathbb{R}^2$

parametrized surface

$$\left.\begin{aligned} T_u &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \\ T_v &= \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \end{aligned}\right\} \text{tangent vectors in } u, v \text{ directions}$$

Example

$$\Phi(\theta, \varphi) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi), \quad \begin{aligned} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{aligned}$$

parametrization of $x^2 + y^2 + z^2 = 9$

$$T_\theta = (-3\sin\varphi\sin\theta, 3\sin\varphi\cos\theta, 0)$$

$$T_\varphi = (3\cos\varphi\cos\theta, 3\cos\varphi\sin\theta, -3\sin\varphi)$$

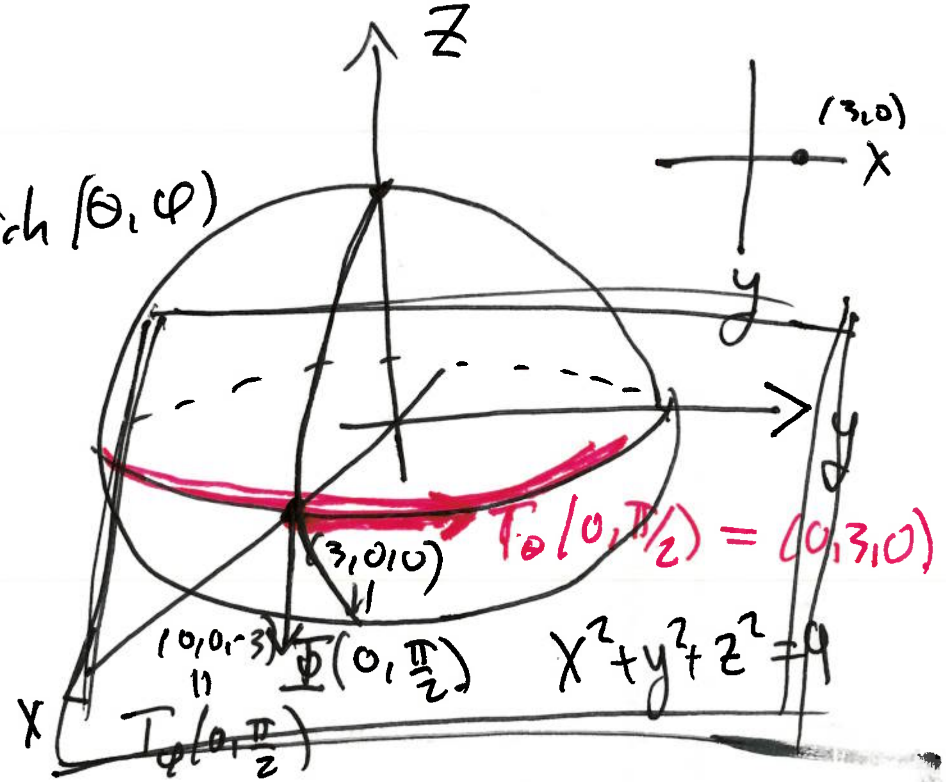
$\Phi(\theta, \varphi) = (3, 0, 0)$

$3\cos(\varphi) = 0$

$\cos(\varphi) = 0$

$\varphi = \frac{\pi}{2}$

What are the $\Phi(\theta, \varphi)$ tangent vectors at point $(3,0,0)$?



$(3,0,0) = \Phi(\theta, \varphi)$ for which (θ, φ)

$$\varphi = \frac{\pi}{2}, \quad \theta = 0$$

$$\Phi(0,\tfrac{\pi}{2}) = (3\sin(\tfrac{\pi}{2})\cos(0),\ 3\sin(\tfrac{\pi}{2})\sin(0),\ 3\cos(\tfrac{\pi}{2}))$$

$$= (3,0,0).$$

Tangent vectors: $T_\theta(0,\frac{\pi}{2}) = (-3\sin(\frac{\pi}{2})\sin(0), 3\sin(\frac{\pi}{2})\cos(0), 0)$

$$= (0,3,0)$$

$$T_\varphi(0,\tfrac{\pi}{2}) = (3\cos(\tfrac{\pi}{2})\cos(0),\ 3\cos(\tfrac{\pi}{2})\sin(0),\ -3\sin(\tfrac{\pi}{2})) = (0,0,-3)$$

Question: What is the normal vector to a surface in terms of $\Phi(u,v)$? What can the normal vector tell us about the surface?

(Hint: Think about tangent planes.)

"The" normal vector is orthogonal to the tangent plane to the surface at the point in question.

Tangent Plane

$\Phi(u,v)$, $(u,v) \in D$ parametrization of a surface S

$T_u \times T_v$ = normal vector to S determined by $\Phi(u,v)$

$\Phi(u_0, v_0) = (x_0, y_0, z_0)$ point on S

$T_u(u_0,v_0) \times T_v(u_0,v_0) = (n_1, n_2, n_3)$ normal vector to S at $\Phi(u_0,v_0) = (x_0,y_0,z_0)$

Equation of tangent plane to S at $\Phi(u_0,v_0) = (x_0,y_0,z_0)$ is

$$(x-x_0,\ y-y_0,\ z-z_0)\cdot\big(T_u(u_0,v_0)\times T_v(u_0,v_0)\big) = 0$$

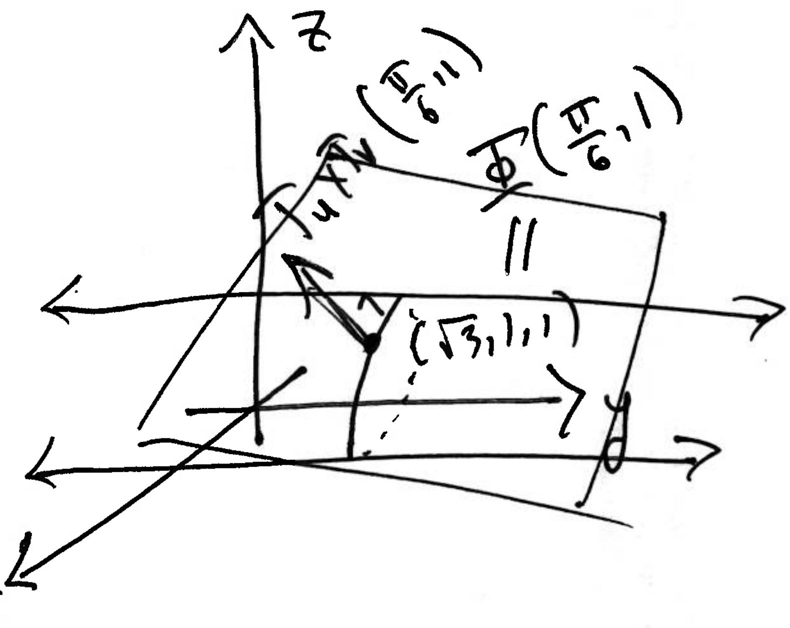
or

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0$$

Fine Print: Tangent plane only exists if $\cancel{T_u(u_0,v_0)\times T_v(u_0,v_0) \neq (0,0,0)}$, $T_u(u_0,v_0)\times T_v(u_0,v_0) \neq (0,0,0)$.

Example

Parametrize cylinder $x^2+z^2=4$. Use parametrization to determine tangent plane at $(\sqrt{3},1,1)$.



Cylindrical coordinates

$x = r\cos\theta$, $u=\theta$

~~y~~ $v=y$

$z = r\sin\theta$ $r=2$

$y=y$

$x^2+z^2=4$ tells us $r=2$

$$\Phi(u,v) = (2\cos u, v, 2\sin u)$$

$0 \le u \le 2\pi$

$-\infty < v < \infty$

$$T_u = (-2\sin u, 0, 2\cos u)$$

$$T_v = (0, 1, 0)$$

$$T_u \times T_v = \det\begin{pmatrix} i & j & k \\ -2\sin u & 0 & 2\cos u \\ 0 & 1 & 0 \end{pmatrix}.$$

$$= (-2\cos u, -0, -2\sin u) = (-2\cos u, 0, -2\sin u)$$

$$\| \; T_u \times T_v$$

$$(\sqrt{3}, 1, 1) = \Phi(u, v)$$

$$(\sqrt{3}, 1, 1) = (2\cos u, v, 2\sin u)$$

$\sqrt{3} = 2\cos u.$ $\qquad \cos u = \frac{\sqrt{3}}{2}$ $\qquad u = \frac{\pi}{6}$

$1 = v$ $\qquad \sin u = \frac{1}{2}$

$1 = 2\sin u$

$$T_u \times T_v\left(\tfrac{\pi}{6}, 1\right) = \left(-2\cos\left(\tfrac{\pi}{6}\right), 0, -2\sin\left(\tfrac{\pi}{6}\right)\right)$$

$$= \boxed{\cancel{(-1, 0, -\sqrt{3})}} = (-\sqrt{3}, 0, -1)$$

normal vector to cylinder at

$$\Phi\left(\tfrac{\pi}{6}, 1\right) = (\sqrt{3}, 1, 1)$$

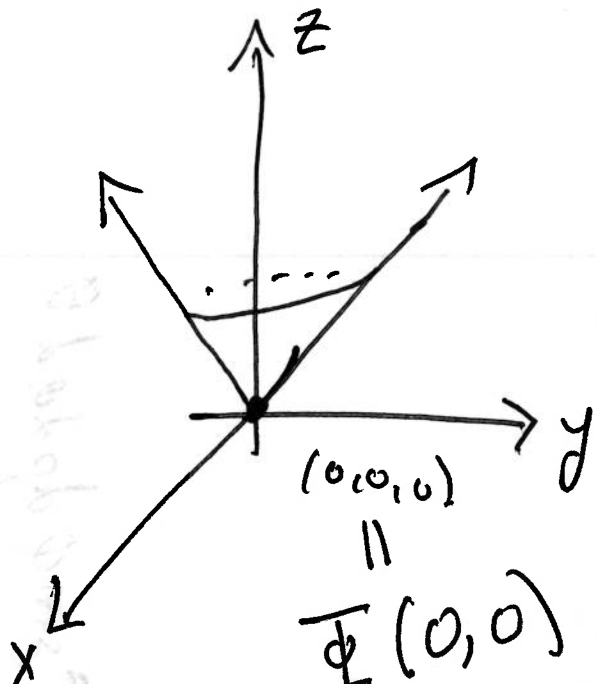
$$(x-\sqrt{3}, y-1, z-1) \cdot (\sqrt{3}, 0, -1) = 0$$

$$\boxed{-\sqrt{3}(x-\sqrt{3}) - (z-1) = 0}$$

__Def__: We say the surface S parametrized by $\Phi(u,v)$ is __regular__ or __smooth__ at $\Phi(u_0, v_0)$ if $T_u(u_0, v_0) \times T_v(u_0, v) \neq (0,0,0)$.

Example

$z = \sqrt{x^2+y^2}$ cone



$\shortparallel$

$\Phi(0,0)$

not smooth only at point $(0,0,0) = \Phi(0,0)$.

Parametrization:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

$\Phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r)$

where is the cone not smooth?

$T_r = (\cos\theta,\ \sin\theta,\ 1)$

$T_\theta = (-r\sin\theta,\ r\cos\theta,\ 0)$

$T_r \times T_\theta = (-r\cos\theta,\ -r\sin\theta,\ r)$ normal vector.

$(0,0,0) = (-r\cos\theta,\ -r\sin\theta,\ r)$ if $r=0$