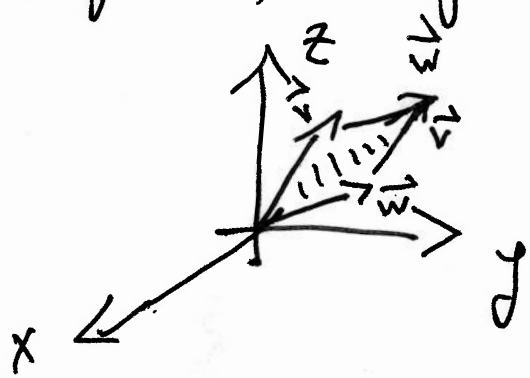
7.4 Area of a Surface

Question: Given $\Phi(u,v)$, (u,v) in D, a parametrization of a surface S, what's the surface area of S?

Ingredients for the answer:

① surface area of parallelogram formed by $\vec{v}, \vec{w}$:

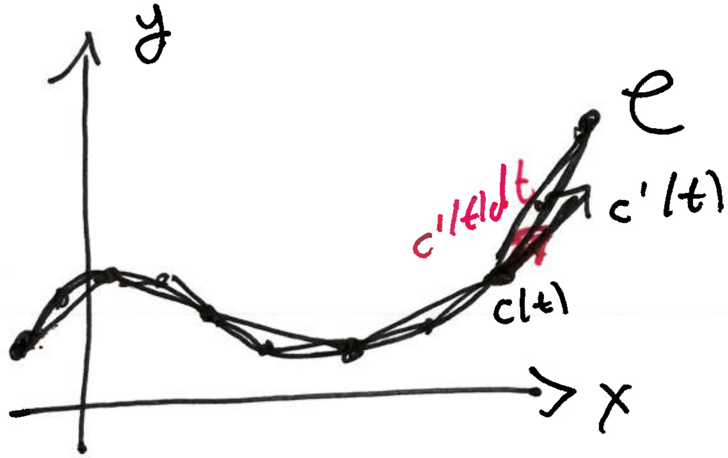


Surface area is $\|\vec{v} \times \vec{w}\|$

② Idea of arc length integral:

$c(t)$, $a \le t \le b$ parametrization of $\mathcal{C}$

Arc length of $\mathcal{C} = \int_a^b \|c'(t)\|\,dt$



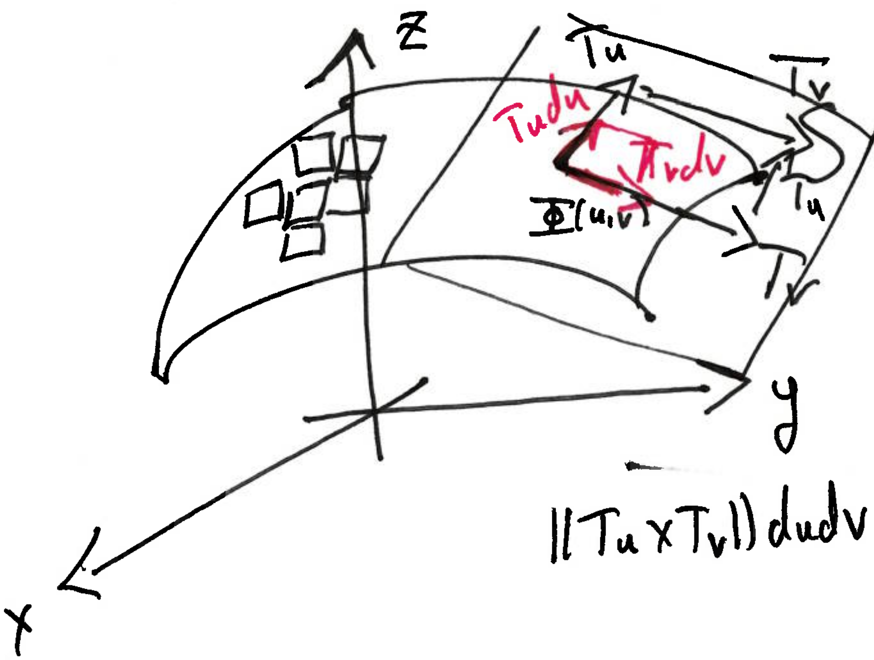
$c'(t)$ approximates $c(t)$

$c'(t)dt$ infinitesimal approximation of $c(t)$

$\|c'(t)\|dt$ = length of infinitesimal piece of $\mathcal{C}$ at $c(t)$

$$c(t) = (x(t), y(t), z(t))$$
$$c'(t) = (x'(t), y'(t), z'(t))$$

$\Phi(u,v)$, (u,v) in D parametrization of surface S



tangent plane ↙ approximates S near $\Phi(u,v)$

$\|T_u \times T_v\|$ = area of parallelogram in tangent plane

$\|T_u \times T_v\| du dv$ = area of infinitesimal parallelogram tangent to S at $\Phi(u,v)$

Def: Area of a Parametrized Surface: The surface area, $A(S)$, of the parametrized surface S is

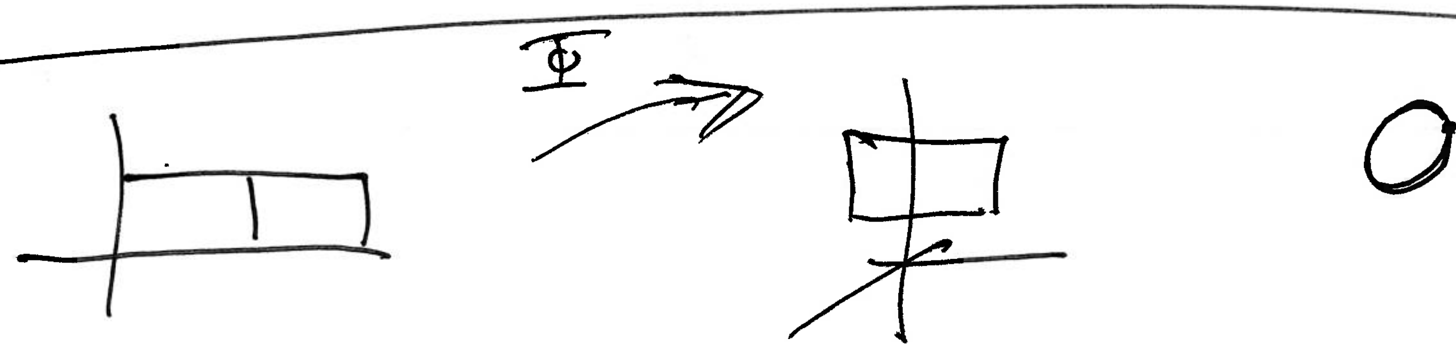
$$A(S) = \iint_D \| T_u \times T_v \| \, du\,dv$$

Fine Print: (1) Assuming D is elementary region.

(2) Φ is of class C^1 and Φ is one-to-one except possibly on boundary of D

(3) S is smooth/regular except at finitely many points.

(4) If S is union of surfaces S_i then area of S is obtained by summing areas of S_i.



Examples

(1) Surface area of cone of height 1 and radius 1.

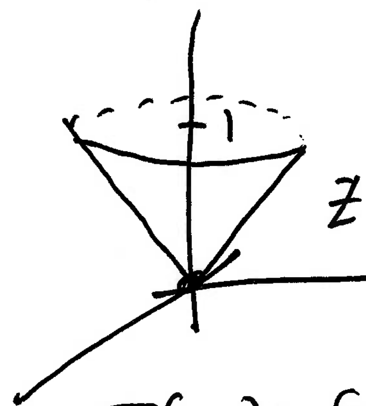
Parametrize using cylindrical:

$x = r\cos\theta$

$y = r\sin\theta$, $x^2+y^2=r^2$

$z = z$

$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$



$z = \sqrt{x^2+y^2}$

$\Phi(x,y) = (x, y, \sqrt{x^2+y^2})$

$D: x^2+y^2 \le 1$

one parametrization

$\Phi(r,\theta) = (x(r,\theta), y(r,\theta), z(r,\theta))$

$\Phi(r,\theta) = (r\cos\theta, r\overset{\sin\theta}{\cancel{\cos\theta}}, r)$, $D: 0 \le \theta \le 2\pi$, $0 \le r \le 1$

$T_r = (\cos\theta, \sin\theta, 1)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$$T_r \times T_\theta = \det \begin{pmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix}.$$

$$= \left(0\cdot\sin\theta - 1\cdot r\cos\theta,\ -(0\cdot\cos\theta - -r\sin\theta),\ r\cos^2\theta - -r\sin^2\theta\right)$$

$$= (-r\cos\theta,\ -r\sin\theta,\ r)$$

$$\|T_r \times T_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$$

$$= \sqrt{r^2 + r^2}$$

$$= \sqrt{2r^2}$$

$$= \sqrt{2}\, r$$

$$\iint_D \|T_r \times T_\theta\|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta = 2\pi \cdot \frac{\sqrt{2}r^2}{2}\Big|_0^1$$

$$= \boxed{\sqrt{2}\pi}$$

D: $0 \le \theta \le 2\pi$
$0 \le r \le 1$

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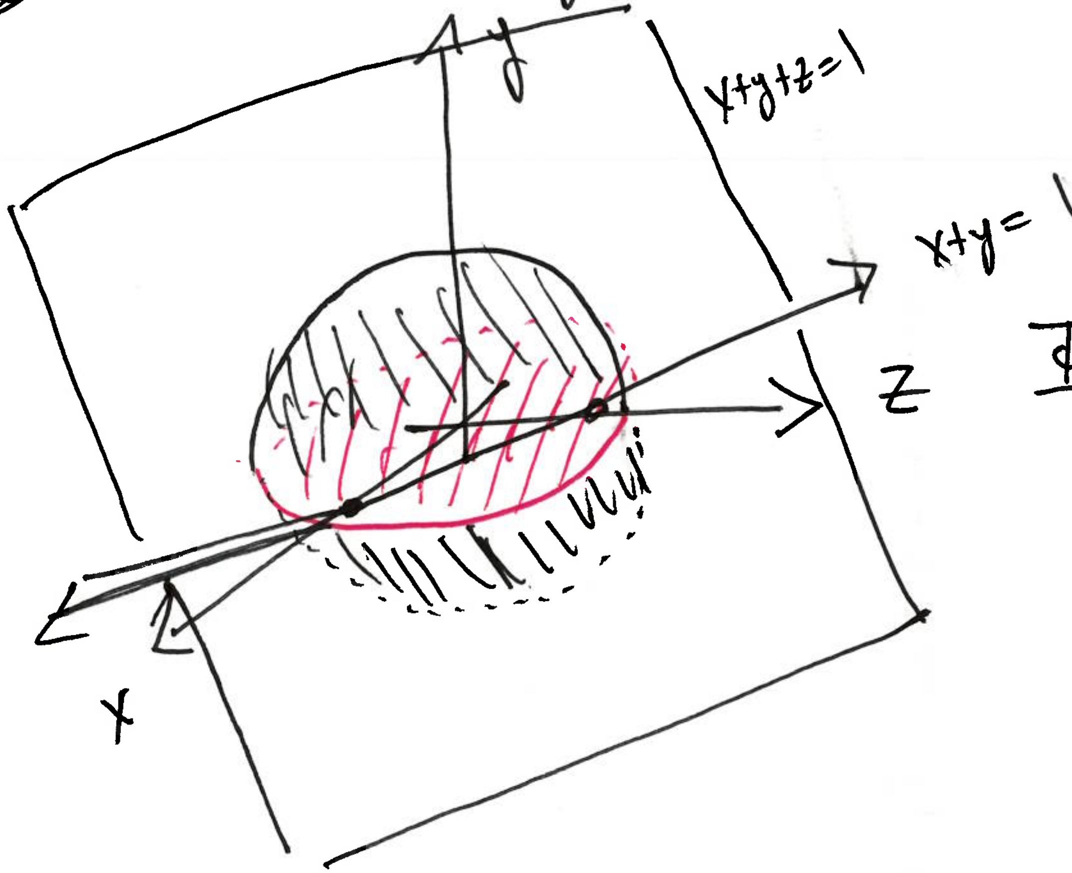
② Find area of surface defined by $x+y+z=1$, $2x^2+3y^2 \leq 1$



$z = 1-x-y$

$\cancel{\Phi(u,v) = \left(\frac{1}{2}\cos u, \frac{1}{3}\sin u, \dots\right.}$

$\Phi(u,v) = \left(\frac{v}{2}\cos u, \frac{v}{3}\sin u, 1-\frac{v}{2}\cos u-\frac{v}{3}\sin u\right)$

$0 \leq u \leq 2\pi$

$0 \leq v \leq 1$

Parametrize $x+y+z=1$: Solve for z, $z = 1-x-y$

$$\Phi(x,y) = (x, y, 1-x-y)$$

$$D: 2x^2+3y^2 \leq 1$$

$$T_x = (1, 0, -1) \qquad T_x \times T_y = \det \begin{pmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$T_y = (0, 1, -1)$$

$$T_x \times T_y = (1, +1, 1)$$

$$\| T_x \times T_y \| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\iint_D \| T_x \times T_y \| \, dx\,dy = \iint_D \sqrt{3} \, dx\,dy, \qquad D: 2x^2 + 3y^2 \leq 1$$

$$x = \frac{1}{\sqrt{2}} r\cos\theta, \quad 0 \leq r \leq 1$$

$$y = \frac{1}{\sqrt{3}} r\sin\theta, \quad 0 \leq \theta \leq 2\pi$$

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \det \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\theta & -\frac{1}{\sqrt{2}} r\sin\theta \\ \frac{1}{\sqrt{3}}\sin\theta & \frac{1}{\sqrt{3}} r\cos\theta \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} r$$

$$\sqrt{3} \int_0^{2\pi} \int_0^1 \frac{r}{\sqrt{6}} \, dr\,d\theta$$

$$\frac{\sqrt{3}}{\sqrt{6}} \cdot 2\pi \frac{1}{2} = \boxed{\frac{\pi}{\sqrt{2}}}$$

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