7.5 Integrals of Scalar Functions Over Surfaces

Questions: S - surface in $\mathbb{R}^3$, $t(x,y,z)$ = temperature at point (x,y,z)

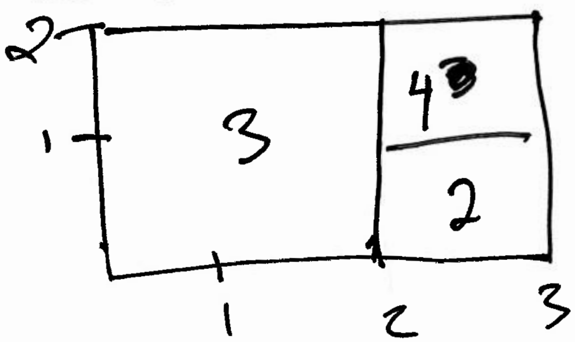
what is the average temperature on S?

S - metal surface, $d(x,y,z)$ = density of S at (x,y,z)

what's the mass of S?

Answer: Integrate $t(x,y,z)$ or $d(x,y,z)$ over S.

Example



S

mass = ?

densities

$$\text{mass} = 2\cdot 1 + 4\cdot 1 + 3\cdot 4 = 18$$

areas

Def: $f(x,y,z)$ = real-valued (scalar-valued) function

S - surface parametrized by $\Phi(u,v)$, $(u,v) \in D$.

The <u>integral of f over S</u> is

$$\iint_S f\, dS = \iint_D f(\Phi(u,v)) \underbrace{\|T_u \times T_v\|\, du\, dv}_{\text{area of infinitesimal piece of } S \text{ at } \Phi(u,v)},$$

$$\left(\text{path integral} \quad \int_{c} f\, ds = \int_a^b f(c(t)) \underbrace{\|c'(t)\|\, dt}, \right)$$

<u>Examples</u>

① Evaluate integral of $f(x,y,z) = x+y$ over surface S given by $\Phi(u,v) = (2u\cos v, 2u\sin v, u)$, $u \in [0,4]$ $(0 \le u \le 4)$

$v \in [0,\pi]$ $(0 \le v \le \pi)$

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$$f(\Phi(u,v)) = f(2u\cos v, 2u\sin v, u)$$

$$= 2u\cos v + 2u\sin v$$

$$\begin{array}{lccc} & i & j & k \\ T_u = & (2\cos v, & 2\sin v, & 1) \\ T_v = & (-2u\sin v, & 2u\cos v, & 0) \end{array}$$

$$T_u \times T_v = (-2u\cos v, --2u\sin v, 4u\cos^2 v + 4u\sin^2 v)$$

$$T_u \times T_v = (-2u\cos v, 2u\sin v, 4u)$$

$$\|T_u \times T_v\| = \sqrt{(-2u\cos v)^2 + (2u\sin v)^2 + (4u)^2}$$

$$= \sqrt{4u^2\cos^2 v + 4u^2\sin^2 v + 16u^2}$$

$$= \sqrt{4u^2 + 16u^2} = \sqrt{20u^2} = 2\sqrt{5}u$$

$$\iint_S f\,dS = \iint_D f(\Phi(u,v))\,\|T_u \times T_v\|\,du\,dv = \int_0^{\pi}\int_0^{4} (2u\cos v + 2u\sin v)\cdot 2\sqrt{5}u\,du\,dv$$

$$= 4\sqrt{5}\int_0^{\pi}\int_0^4 u^2\cos v + u^2\sin v \, du\,dv$$

$$= 4\sqrt{5}\left(\int_0^{\pi} \cos v + \sin v \, dv\right) \cdot \frac{u^3}{3}\Big|_0^4$$

$$= \frac{4^4\sqrt{5}}{3}\left(+\sin v - \cos v\Big|_0^{\pi}\right)$$

$$= \frac{4^4\sqrt{5}}{3}\left(- -1 - -1\right)$$

$$= \frac{2\cdot 4^4\sqrt{5}}{3} = \boxed{\frac{512\sqrt{5}}{3}}$$

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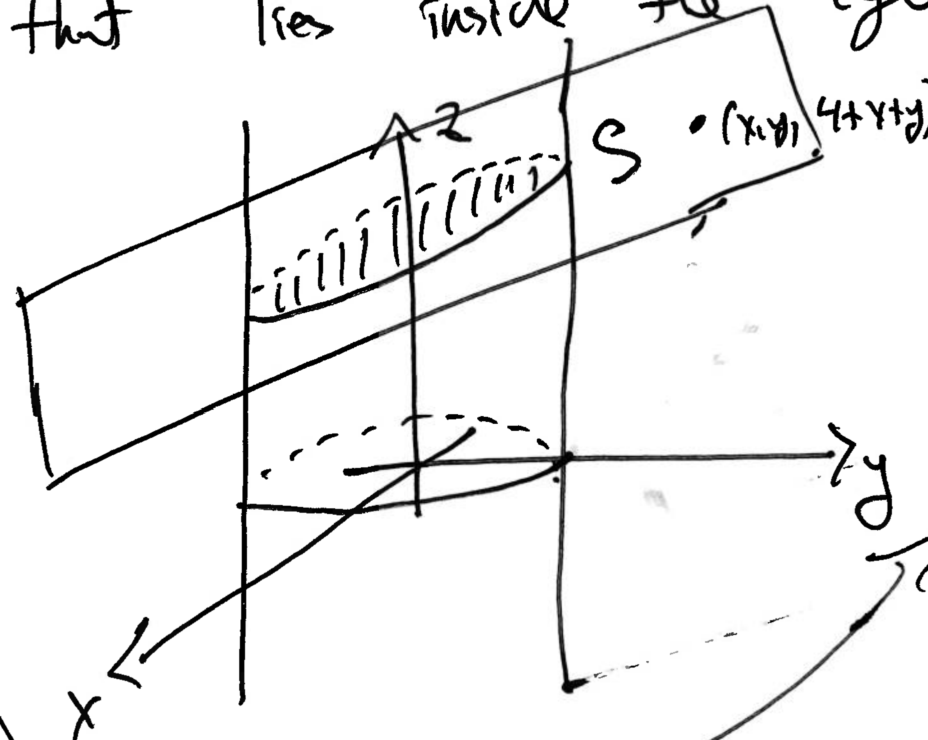
② Evaluate the integral

$$\iint_S x^2 z + y^2 z \, dS$$

where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$.



$\Phi(u,v) = (u, v, 4+u+v)$

$D: u^2 + v^2 \leq 4$

$f(x,y,z) = x^2 z + y^2 z$

$$\begin{array}{cc} & \begin{array}{ccc} i & j & k \end{array} \\ T_u = & (1, 0, 1) \\ T_v = & (0, 1, 1) \end{array}$$

$T_u \times T_v = (-1, -1, 1)$

$\|T_u \times T_v\| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}$

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$$x = u, \quad y = v, \quad z = 4 + u + v$$

$$f(\Phi(u,v)) = f(u, v, 4+u+v)$$

$$= u^2(4+u+v) + v^2(4+u+v)$$

$$\iint_S f\,dS = \iint_D f(\Phi(u,v))\,\|T_u \times T_v\|\,du\,dv$$

$$= \iint_{u^2+v^2 \le 4} \underbrace{\left(u^2(4+u+v) + v^2(4+u+v)\right)}_{(u^2+v^2)(4+u+v)} \cdot \sqrt{3}\,du\,dv$$

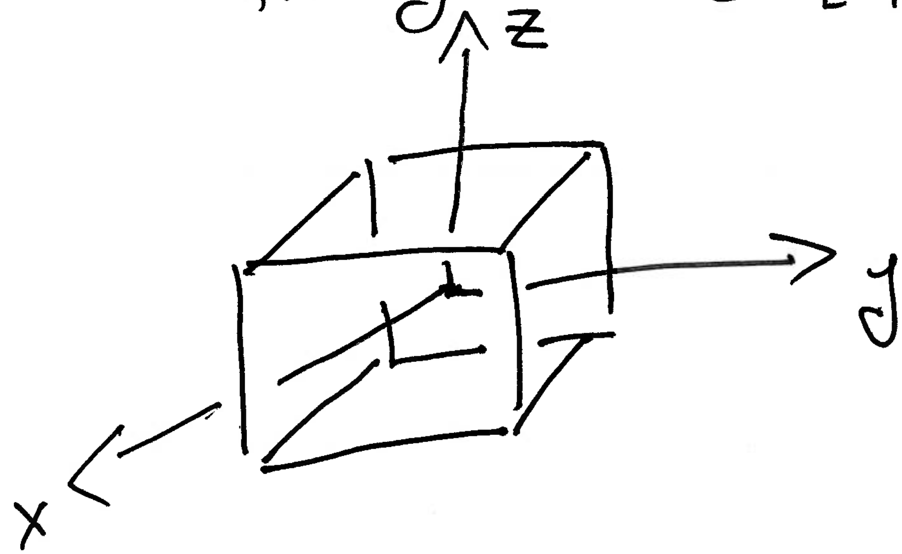
Polar Coord. change of variables

$$u = r\cos\theta \qquad 0 \le \theta \le 2\pi$$
$$v = r\sin\theta \; , \quad 0 \le r \le 2$$
$$u^2 + v^2 = r^2$$

$$= \sqrt{3} \int_0^{2\pi} \int_0^2 r^2 (4 + r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \sqrt{3} \int_0^{2\pi} \int_0^2 4r^3 + r^4\cos\theta + r^4\sin\theta \,dr\,d\theta$$

(3) Parametrize faces of cube $C = [-1,1]\times[-1,1]\times[-1,1]$.



$-1 \le x \le 1$

$-1 \le y \le 1$

$-1 \le z \le 1$

6 faces

front	$x=1$	$\rightsquigarrow \Phi(u,v) = (1, u, v), \quad -1 \le u \le 1, \ -1 \le v \le 1$
back	$x=-1$	$\rightsquigarrow \Phi(u,v) = (-1, u, v) \quad -1 \le u \le 1, \ -1 \le v \le 1$
left	$y=-1$	
right	$y=1$	
top	$z=1$	
bottom	$z=-1$	$\rightsquigarrow \Phi(u,v) = (u, v, -1), \quad -1 \le u \le 1, \ -1 \le v \le 1$

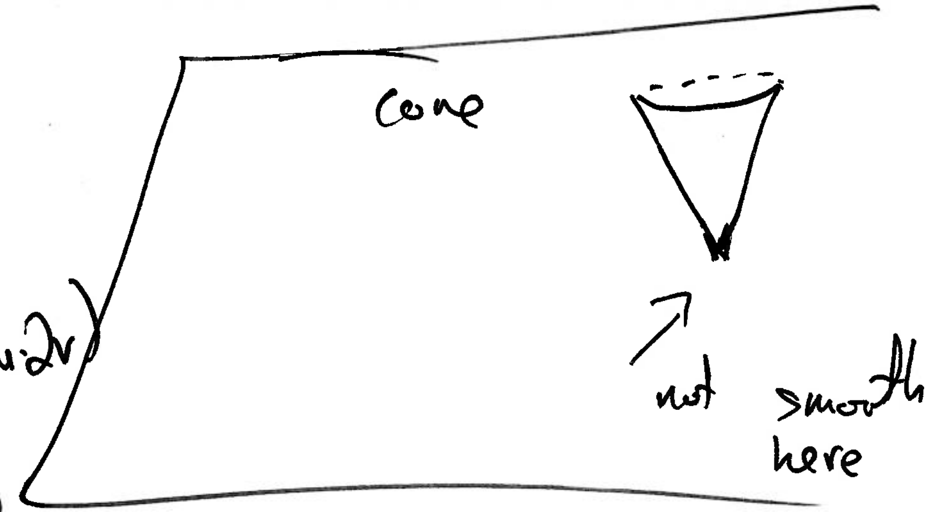
Example

Determine where $\Phi(u,v) = (u^2-v^2, u^2+v^2, v)$ is not smooth.

$T_u = (2u, 2u, 0)$

$T_v = (-2v, 2v, 1)$

$T_u \times T_v = (2u, -2u, 2u\cdot 2v - -2u\cdot 2v)$



$T_u \times T_v = (2u, -2u, 8uv.)$

$\Phi(u,v)$ is not smooth if $(0,0,0) = T_u \times T_v$

$$(0,0,0) = (2u, -2u, 8uv)$$

$$\left.\begin{matrix} 0 = 2u \\ 0 = -2u \end{matrix}\right\} \longrightarrow u = 0$$

$$0 = 8uv \longrightarrow u=0, \ v = \text{anything}$$

$\Phi(u,v)$ is not smooth for the values $(0,v)$, where $-\infty < v < \infty$

or points $\Phi(0,v) = (-v^2, v^2, v)$ where $-\infty < v < \infty$.