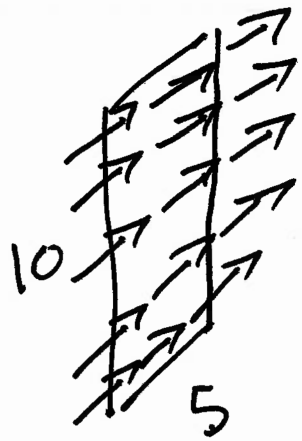
7.6 Surface Integrals of Vector Fields

Motivating Example: Wind is blowing through a window



$F(x,y,z) = \langle 1,2,2 \rangle$ vector field for wind

window is in the $y=0$ plane

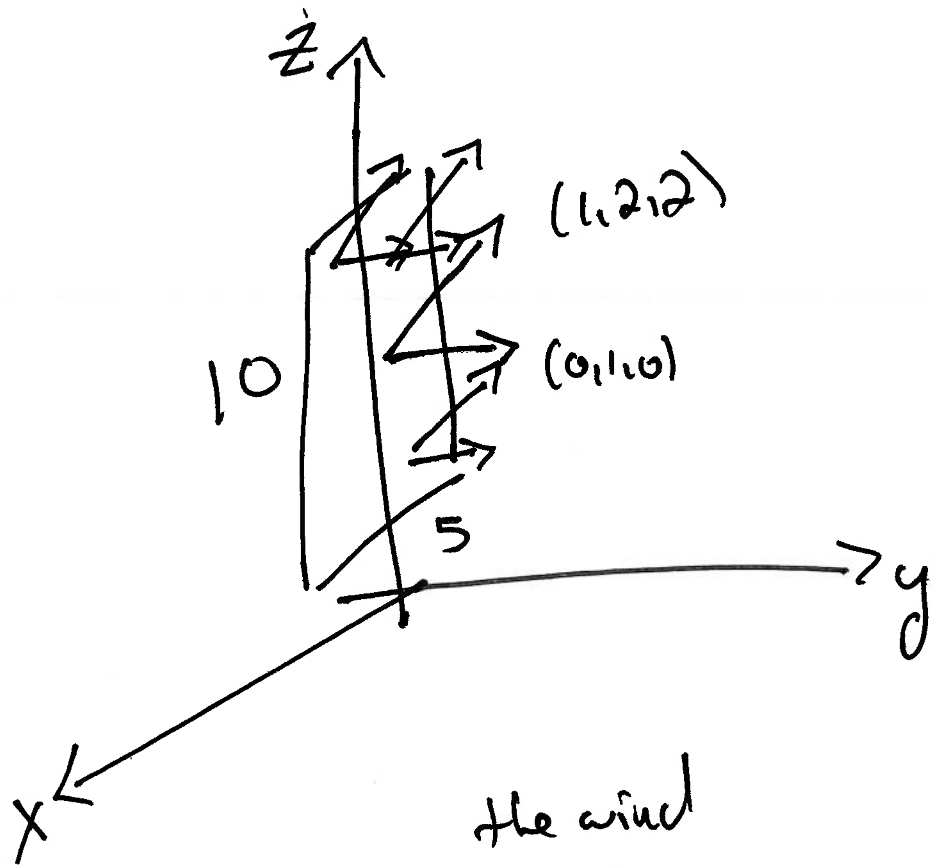
$0 \le x \le 5$, $0 \le z \le 10$

(in xz-plane)

Question: At what rate is the wind flowing through the window?

Another example: F = vector field that is the velocity field of a liquid. Put an imagined surface, S, in liquid. What is the rate at which the fluid (liquid) is crossing S (measured in say cubic meters per second)?

Answer to first question:



The amount that the wind (1,2,2) is going in the direction (0,1,0) of the window is

$$(1,2,2)\cdot(0,1,0) = 1\cdot 0 + 2\cdot 1 + 2\cdot 0 = 2$$

Answer: $\cancel{50}\cdot$ (Area of plane) $\cdot$ (dot product of vectors)

$$\parallel$$

$$50\cdot 2 = 100$$

<u>Answer to second question</u>: The word for what we want to calculate is the <u>flux</u> of F through S and it is calculated by integrating the vector field over the surface.

<u>Def</u>: F = vector field $\quad F = (F_1, F_2, F_3)$

S = surface parametrized by $\Phi(u,v)$, where $(u,v) \in D$

The <u>surface integral of F over</u> Φ is

$$\iint_{\Phi} F \cdot d\vec{S} = \iint_{D} F(\Phi(u,v)) \cdot T_u \times T_v \, du\,dv$$

<u>Example</u>: S: $x^2 + y^2 + z^2 = 1$ unit sphere

$F(x,y,z) = (x,y,z)$ outward pointing vector field

what is the flux of F through S?

$$\Phi(\theta,\varphi) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi), \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$

$$\begin{array}{lcl} & & \quad i \qquad\qquad\qquad j \qquad\qquad k \\ T_\theta & = & (-\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0) \\ T_\varphi & = & (\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi) \end{array}$$

$$T_\theta \times T_\varphi = (-\sin^2\varphi\cos\theta,\ -\sin^2\varphi\sin\theta,\ -\sin\varphi\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta)$$

$$T_\theta \times T_\varphi = (-\sin^2\varphi\cos\theta,\ -\sin^2\varphi\sin\theta,\ -\sin\varphi\cos\varphi)$$

$$F(x,y,z) = (x,y,z) \qquad \iint_D F(\Phi(u,v)) \cdot T_u \times T_v \, du\,dv$$

$$F(\Phi(u,v)) = F(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

$$= (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

$$F(\Phi(u,v)) \cdot T_\theta \times T_\varphi = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot$$

$$\cdot\, (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi)$$

$$= -\sin^3\varphi\cos^2\theta - \sin^3\varphi\sin^2\theta - \sin\varphi\cos^2\varphi$$

$$= -\sin^3\varphi(1) - \sin\varphi\cos^2\varphi$$

$$= -\sin\varphi(\sin^2\varphi + \cos^2\varphi)$$

$$= -\sin\varphi$$

$$\iint_D F(\Phi(\theta,\varphi)) \cdot T_\theta \times T_\varphi \, d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\pi} -\sin\varphi \, d\varphi\, d\theta$$

$$= 2\pi \cdot \cos\varphi \Big|_0^{\pi} = 2\pi(-1-1)$$

$$= \boxed{-4\pi}$$

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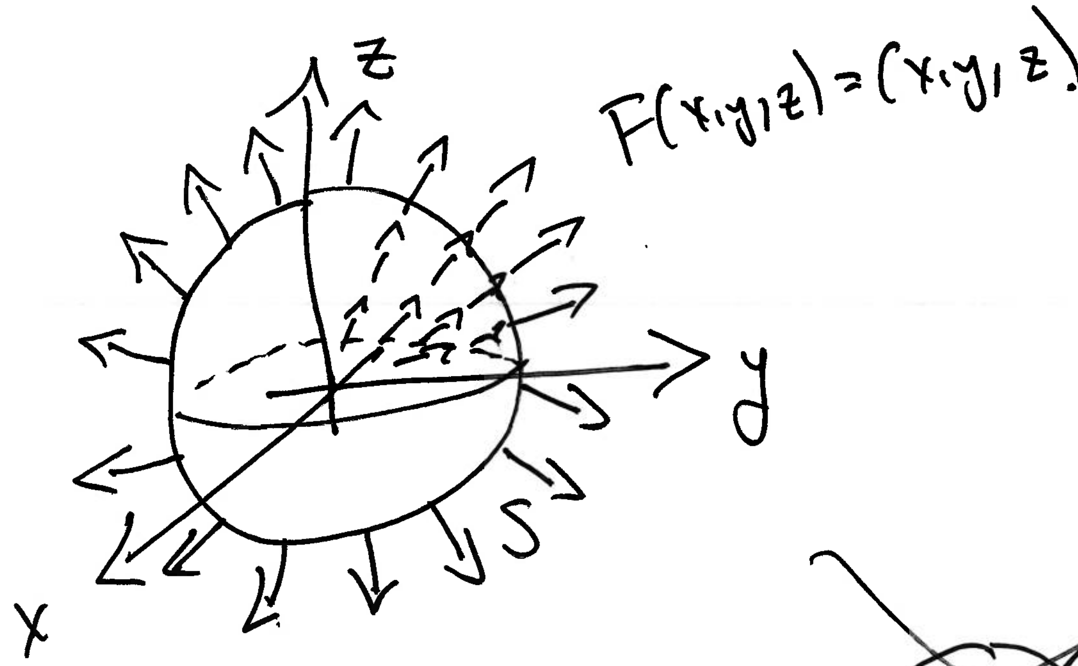
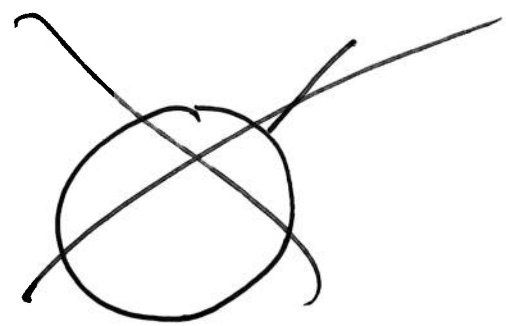
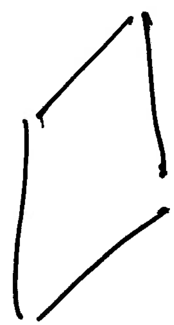
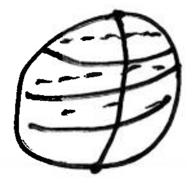
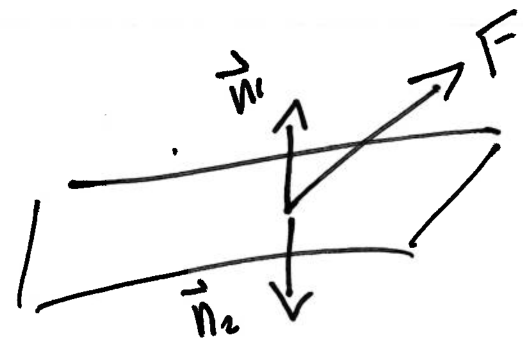


z
$F(x,y,z) = (x,y,z)$
y
x
S









$F \cdot \vec{n}_1 > 0$

$F \cdot \vec{n}_2 < 0$

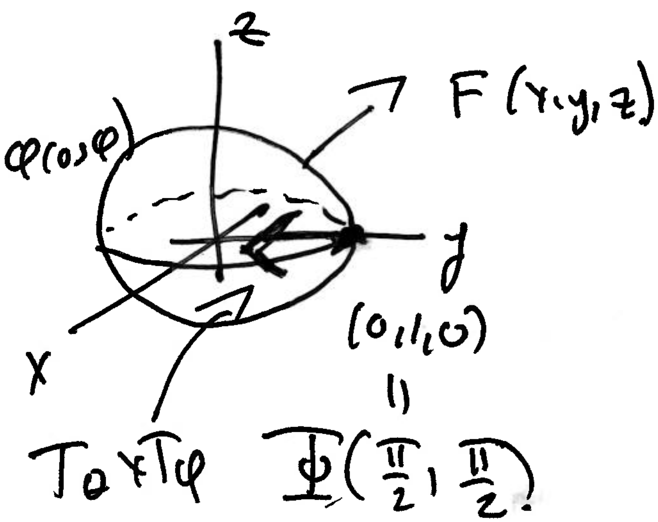
$\vec{n}_1, \vec{n}_2$ normal vectors pointing in opposite directions

Sphere example

$$T_\theta \times T_\varphi = (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi)$$

$$T_\theta \times T_\varphi(\tfrac{\pi}{2}, \tfrac{\pi}{2}) = (-1\cdot 0, -1, -1\cdot 0)$$

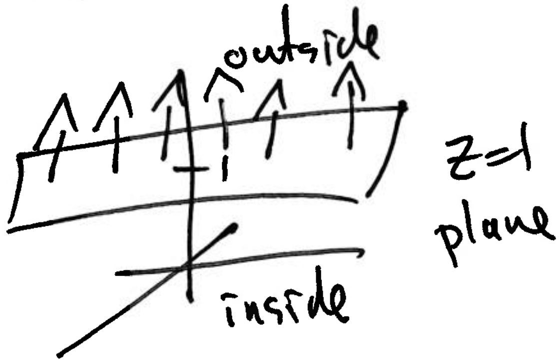
$$= (0, -1, 0)$$



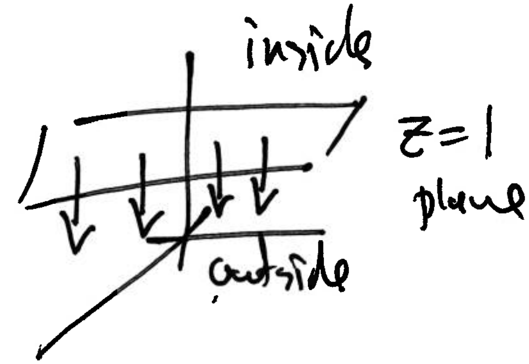
Orientation

An oriented surface is a two-sided surface with one side specified as the outside or positive side. The other side is called the inside or negative side.

Example



orientation of $z=1$ plane



other orientation

$S =$ oriented surface

At each point $(x, y, z) \in S$ there are two unit normal vectors, $\vec{n}_1$, $\vec{n}_2$ where $\vec{n}_1 = -\vec{n}_2$. The orientation of S chooses a unit normal vector at each point of S, it chooses the unit normal vector pointing towards the outside.

$(x,y,z) \in S$, $\quad \vec{n}(x,y,z) =$ unit normal vector determined by orientation

On other hand a parametrization $\Phi(u,v)$ also determines a unit normal vector at each point:

$$\frac{T_u \times T_v}{\| T_u \times T_v \|} = \text{unit normal vector determined by } \Phi$$

If

$$\frac{T_u \times T_v}{\| T_u \times T_v \|} = \vec{n}(\Phi(u,v))$$

then Φ is called <u>orientation preserving</u>.

If

$$\frac{T_u \times T_v}{\| T_u \times T_v \|} = -\vec{n}(\Phi(u,v))$$

then Φ is called orientation reversing.

Examples

Give sphere $x^2+y^2+z^2=1$ the orientation such that the normal vector points away from the origin.

Then $\Phi(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ is orientation reversing because the normal vector determined by $\Phi(\theta,\varphi)$ points towards the origin.