7.6 Surface Integrals of Vector Fields

Motivating Example: Wind is blowing through a window

\[ F(x,y,z) = (1, 2, 2) \] vector field for wind

window is in the y-z plane

\[ 0 \leq x \leq 5, \quad 0 \leq z \leq 10 \]

(in xz-plane)

Question: At what rate is the wind flowing through the window?

Another example: \( F = \) vector field that is the velocity field of a liquid. Put an imagined surface \( S \), in liquid. What is the rate at which the fluid is crossing \( S \) (measured in say cubic meters per second)?
Answer to first question:

The amount that \((1,2,2)\) is going in the direction \((0,1,10)\) is

\[
(1,2,2) \cdot (0,1,10) = 1 \cdot 0 + 2 \cdot 1 + 2 \cdot 10 = 22
\]

Answer: \(50 \cdot 2 = 100\)
Answer to second question: The word for what we want to calculate is the flux of \( F \) through \( S \) and it is calculated by integrating the vector field over the surface.

**Def:** \( F = \) vector field \( F = (F_1, F_2, F_3) \)

\[ S = \text{surface parametrized by } \Phi(u,v), \text{ where } (u,v) \in D \]

The surface integral of \( F \) over \( S \) is

\[ \iint_S F \cdot d\mathbf{S} = \iint_D F(\Phi(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \, du \, dv \]

**Example:** \( S = x^2 + y^2 + z^2 = 1 \) unit sphere \( F(\mathbf{x},y,z) = (x,y,z) \) outward pointing vector field

What is the flux of \( F \) through \( S \)?
\[ \Phi(\theta, \varphi) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi), \quad 0 \leq \theta \leq 2\pi \]

\[ T_\theta = \begin{pmatrix} \sin \varphi \sin \theta & \sin \varphi \cos \theta & 0 \end{pmatrix} \]

\[ T_\varphi = \begin{pmatrix} \cos \varphi & \sin \varphi \sin \theta & -\sin \varphi \cos \varphi \end{pmatrix} \]

\[ T_\theta \times T_\varphi = \begin{pmatrix} -\sin^2 \varphi \cos \theta & -\sin^2 \varphi \sin \theta & -\sin \varphi \cos \varphi \sin \theta \cos \theta \\ -\sin \varphi \cos \varphi \sin \theta & -\sin \varphi \cos \varphi \sin \theta & -\sin \varphi \cos \varphi \sin \theta \cos \theta \end{pmatrix} \]

\[ T_\theta \times T_\varphi = \begin{pmatrix} -\sin^2 \varphi \cos \theta, & -\sin^2 \varphi \sin \theta, & -\sin \varphi \cos \varphi \end{pmatrix} \]

\[ F(x, y, z) = \int \int_D F(\Phi(u, v)) \cdot T_\nu \times T_\nu \, du \, dv \]

\[ F(\Phi(u, v)) = F(\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi) \]

\[ = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi) \]
\[ F(\varphi(\theta, \phi)) \cdot T_\theta \times T_\phi = (\sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \cos \phi) \]
\[ \times (-\sin^2 \phi \cos \theta, -\sin^2 \phi \sin \theta, -\sin \phi \cos \phi) \]
\[ = -\sin^3 \phi \cos^2 \theta - \sin^3 \phi \sin^2 \theta - \sin \phi \cos^2 \phi \]
\[ = -\sin^3 \phi (1) - \sin \phi \cos^2 \phi \]
\[ = -\sin \phi (\sin^2 \phi + \cos^2 \phi) \]
\[ = -\sin \phi \]

\[ \iiint_{D} F(\varphi(\theta, \phi)) \cdot T_\theta \times T_\phi \, d\theta \, d\phi \, d\phi = \int_{0}^{2\pi} \int_{0}^{\pi} -\sin \phi \, d\phi \, d\theta \]
\[ = \left[ 2\pi \cdot \cos \phi \right]_{0}^{\pi} = 2\pi (1 - 1) \]
\[ = [-4\pi] \]
\[ F(x, y, z) = (x, y, z) \]
Sphere example

\[ \mathbf{T}_0 \times \mathbf{T}_\phi = (-\sin^2 \phi \cos \theta, -\sin^2 \phi \sin \theta, -\sin \phi \cos \phi) \]

\[ \mathbf{T}_0 \times \mathbf{T}_\phi \left( \frac{\pi}{2}, \frac{\pi}{2} \right) = (-1, 0, -1, 0) \]

\[ = (0, -1, 0) \]
Orientation

An oriented surface is a two-sided surface with one side specified as the \textit{outside} or \textit{positive} side. The other side is called the \textit{inside} or \textit{negative} side.

Example

\begin{itemize}
  \item Orientation of \( z=1 \) plane
  \item At each point \((x,y,z) \in S\) there are two unit normal
vectors, \( \vec{n}_1, \vec{n}_2 \) where \( \vec{n}_1 = -\vec{n}_2 \). The orientation of \( S \) chooses a unit normal vector at each point of \( S \), it chooses the unit normal vector pointing towards the outside.

\[(x_1 y_1 z) \in S, \quad \vec{n}(x_1 y_1 z) = \text{unit normal vector determined by orientation}\]

On other hand a parametrization \( \Phi(u, v) \) also determines a unit normal vector at each point:

\[\frac{T_u \times T_v}{||T_u \times T_v||} = \text{unit normal vector determined by } \Phi(u, v)\]

If

\[\frac{T_u \times T_v}{||T_u \times T_v||} = \vec{n}(\Phi(u, v))\]

then \( \Phi \) is called orientation preserving.
If \[ \frac{T_u \times T_v}{\|T_u \times T_v\|} = -\hat{n} (\Phi(u,v)) \]

then \( \Phi \) is called orientation reversing.

**Example**

Given sphere \( x^2 + y^2 + z^2 = 1 \) the orientation such that the normal vector points away from the origin.

Then \( \Phi(\theta, \phi) = (\infty \sin \phi \cos \theta, \infty \sin \phi \sin \theta, \cos \phi) \)

is orientation reversing because the normal vector determined by \( \Phi(\theta, \phi) \) points towards the origin.