Homework 6 due Wed. 11/13 3:50pm

Surface Integrals of vector fields

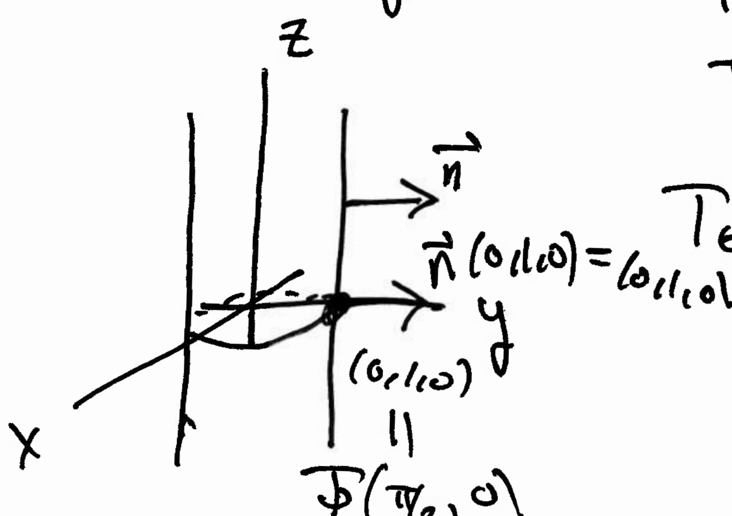
Example: $S: x^2+y^2=1$ cylinder with orientation given by the normal pointing away from the origin.

Is $\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$ orientation preserving or reversing?

$0 \le \theta \le 2\pi,\ -\infty < z < \infty$

$$\begin{array}{cccc} & i & j & k \\ T_\theta = (& -\sin\theta, & \cos\theta, & 0) \\ T_z = (& 0, & 0, & 1) \end{array}$$

$$T_\theta \times T_z = (\cos\theta, \sin\theta, 0)$$



$(0,1,0) = \Phi(\theta, z)$

$(0,1,0) = (\cos\theta, \sin\theta, z)$

$0 = \cos\theta,\ 1 = \sin\theta,\ z = 0$

$\theta = \frac{\pi}{2}$

$$T_\theta \times T_z(\pi/2, 0) = (\cos(\tfrac{\pi}{2}), \sin(\tfrac{\pi}{2}), 0) = (0,1,0)$$

Φ is orientation preserving because

$$\underbrace{T_\theta \times T_z\left(\tfrac{\pi}{2}, 0\right)}_{\text{parametrization}} = (0,1,0) = \underbrace{\vec{n}(0,1,0)}_{\text{orientation.}}$$

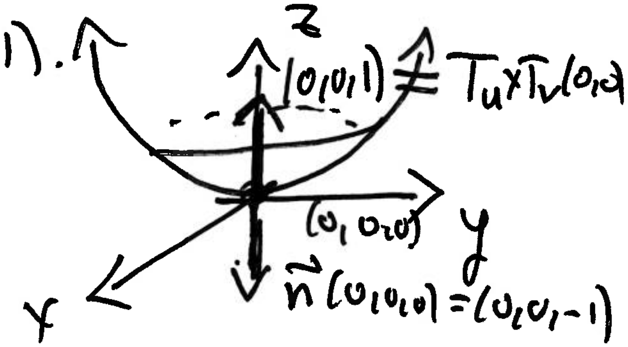
Thm: S - surface, F - vector field, S has an orientation.
Φ_1, Φ_2 are two parametrizations of S.
If Φ_1, Φ_2 are orientation preserving

$$\iint_{\Phi_1} F \cdot d\vec{S} = \iint_{\Phi_2} F \cdot d\vec{S}.$$

If Φ_1 is orientation preserving and Φ_2 is orientation reversing, then

$$\iint_{\Phi_1} F \cdot d\vec{S} = -\iint_{\Phi_2} F \cdot d\vec{S}.$$

Example: S = surface given by $z = x^2 + y^2$, $0 \le x \le 2$, $0 \le y \le 2$ with orientation such that the normal vector at $(0,0,0)$ is $(0,0,-1)$. Let $F(x,y,z) = (y, -x, 1)$.



Calculate

$$\iint_S F \cdot d\vec{S}.$$

① Parametrize S:

$$\Phi(u,v) = (u, v, u^2 + v^2) \qquad \left(\begin{matrix} x = u \\ y = v \end{matrix},\ z = u^2 + v^2 \right).$$

$$D: \begin{matrix} 0 \le u \le 2 \\ 0 \le v \le 2 \end{matrix}$$

② Check orientation reversing or preserving:

$$\begin{array}{ccc} & \hat{i} \quad \hat{j} \quad k & \\ T_u = & (1, 0, 2u) & \\ T_v = & (0, 1, 2v) & \end{array}$$

$$T_u \times T_v = (-2u, -2v, 1),$$

$$T_u \times T_v(0,0) = (0,0,1)$$

$(0,0,0) = \Phi(u,v)$, $(0,0,0) = (u, v, u^2+v^2)$
$u = 0,\ v = 0$

$\boxed{\text{orientation reversing.}}$

(3) Calculate integral and negate answer because orientation ~~reserva~~ reversing.

$F(x,y,z) = (y, -x, 1)$

$$\iint_D F(\Phi(u,v)) \cdot T_u \times T_v \, du\,dv,$$

$$F(\Phi(u,v)) = F(u, v, u^2+v^2) = (v, -u, 1)$$

$x = u, \; y = v, \; z = u^2 + v^2$

$$\int_0^2 \int_0^2 (v, -u, 1) \cdot (-2u, 2v, 1) \, du\,dv =$$

$$= \int_0^2 \int_0^2 -2uv + 2uv + 1 \, du\,dv$$

$$= \int_0^2 \int_0^2 1 \cdot du\,dv = 2 \cdot 2 = 4.$$

$$\boxed{\text{Answer} = -4}$$

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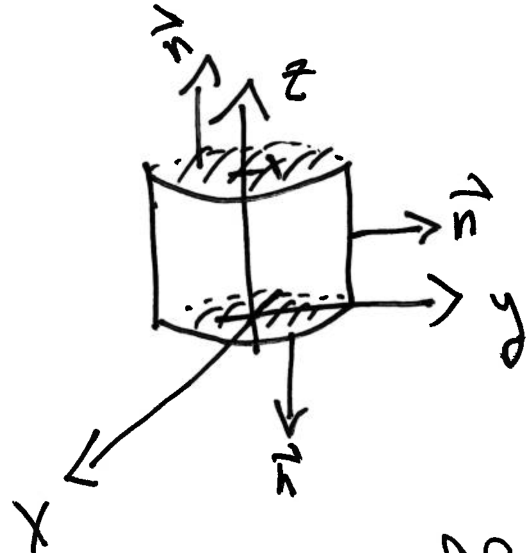
Example

Evaluate $\iint_S F \cdot d\vec{S}$ where $F(x,y,z) = (1, 1, z(x^2+y^2))$ and S is the surface of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 1$, and S is oriented with normal vector pointing away from the origin.



Three surfaces. (so 3 integrals)

cylinder: $x^2+y^2=1$, $0 \le z \le 1$ — S_1

top: $z=1$, $x^2+y^2 \le 1$ — S_2

bottom: $z=0$, $x^2+y^2 \le 1$ — S_3

$$\iint_S F \cdot d\vec{S} = \iint_{S_1} F \cdot d\vec{S} + \iint_{S_2} F \cdot d\vec{S} + \iint_{S_3} F \cdot d\vec{S}$$

S_1 cylinder $x^2+y^2=1$, $0 \le z \le 1$, $F(x,y,z) = (1,1,z(x^2+y^2))$

$$\Phi(u,v) = (\cos(u), \sin(u), v), \qquad \begin{aligned} x &= \cos(u) \\ y &= \sin(u) \\ z &= v \end{aligned} \qquad D: \begin{aligned} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{aligned}$$

$\Phi(u,v)$ is orientation preserving because this is the first example from class today.

$$(1,1,v(\cos^2(u)+\sin^2(u))) = (1,1,v)$$

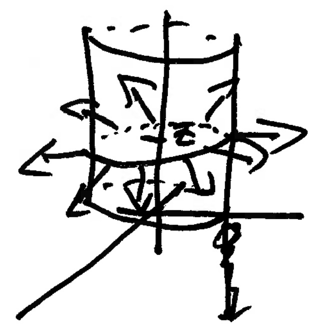
$$T_u \times T_v = (\cos u, \sin u, 0), \qquad F(\Phi(u,v)) = F(\cos(u), \sin(u), v).$$

$$\iint_{S_1} F \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 F(\Phi(u,v)) \cdot T_u \times T_v \, dv\, du$$

$$= \int_0^{2\pi}\int_0^1 (1,1,v) \cdot (\cos u, \sin u, 0)\, dv\, du$$

$$= \int_0^{2\pi}\int_0^1 \cos u + \sin u \, dv\, du$$

$$= 1 \cdot \left(+\sin u - \cos u \Big|_0^{2\pi} \right) = 0$$



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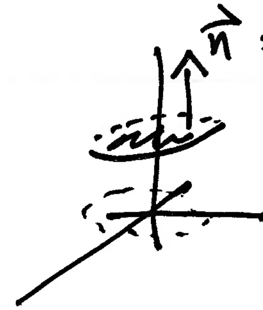
S_2 top $z=1$, $x^2+y^2 \le 1$, $F(x,y,z) = (1,1,z(x^2+y^2))$

$\Phi(u,v) = (u,v,1)$, $u^2+v^2 \le 1$



$D \circlearrowleft$

$$\begin{array}{ccc} i & j & k \end{array}$$

$T_u = (1,0,0)$

$T_v = (0,1,0)$

$T_u \times T_v = (0,0,1)$ orientation preserving since

$T_u \times T_v = (0,0,1) = \vec{n}$

$F(\Phi(u,v)) = F(u,v,1) = (1,1,1(u^2+v^2))$

$x=u, y=v, z=1$

$$\iint_D F(\Phi(u,v)) \cdot T_u \times T_v \, du\,dv = \iint_{u^2+v^2 \le 1} (1,1,u^2+v^2) \cdot (0,0,1) \, du\,dv$$

$$= \iint_{u^2+v^2 \le 1} u^2+v^2 \, du\,dv$$

Polar: $u = r\cos\theta$, $v = r\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$, $u^2+v^2 = r^2$, $du\,dv = r\,dr\,d\theta$

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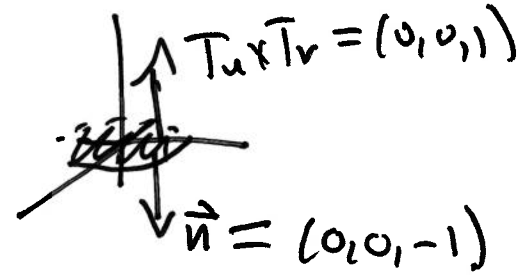
$$= \int_0^{2\pi} \int_0^1 r^2 \, r\,dr\,d\theta$$

$$= 2\pi \cdot \frac{r^4}{4}\Big|_0^1 = \frac{2\pi}{4} = \frac{\pi}{2}$$

S_3 = bottom $z=0$, $x^2+y^2 \le 1$, $F(x,y,z) = (1,1,z(x^2+y^2))$

$\Phi(u,v) = (u,v,0)$, $u^2+v^2 \le 1$



$T_u = (1,0,0)$

$T_v = (0,1,0)$

$T_u \times T_v = (0,0,1)$

$\boxed{\begin{array}{c} T_u \times T_v = (0,0,1) = -\vec{n} \\ \text{orientation reversing} \end{array}}$

$F(\Phi(u,v)) = F(u,v,0) = (1,1,0)$

$x=u,\ y=v,\ z=0$

$$\iint_D F(\Phi(u,v)) \cdot T_u \times T_v \,du\,dv = \iint_D (1,1,0) \cdot (0,0,1) = 0$$

Answer: $0 + \frac{\pi}{2} + 0 = \boxed{\cancel{\frac{\pi}{3}}}$ $\boxed{\frac{\pi}{2}}$

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