Midterm on Friday

- no calculators or electronic aids
- one page (front and back 8.5x11) of <u>hand-written</u> notes
- emphasis on chapter 7 (7.1 - 7.6)
- 5 problems (no extra credit)
- Topics:
  - path ~~line~~ integrals (scalar function over curve)
  - line integrals (vector field over curve)
  - surface integrals of scalar functions
  - " " vector fields
  - properties of curves: parametrizations, orientation
  - properties of surfaces: parametrizations, normal vectors, tangent vectors, smooth/normal, equation of tangent planes, orientation

8.1 Green's Theorem

FTOC for single variable calculus

$$\int_a^b \frac{dF}{dx}(x)\,dx = F(b) - F(a)$$

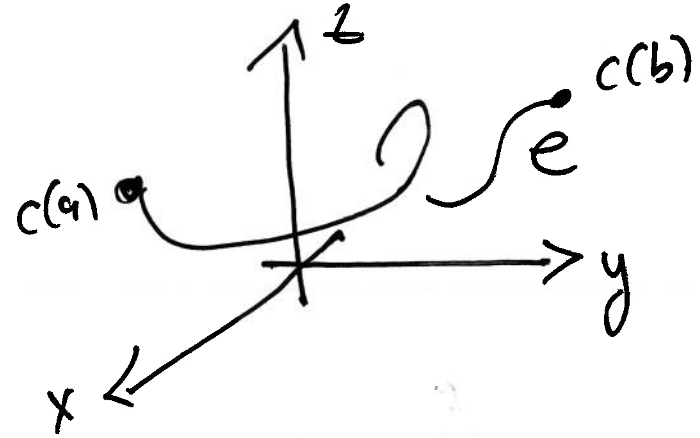
$[a,b]$ is an interval 

endpoints are a and b (which are the boundary of $[a,b]$).

FTOC for gradient functions

$c(t)$, $a \le t \le b$ parametrization of curve $\mathcal{C}$

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$$



$\mathcal{C}$ is curve with endpoints $c(a)$ and $c(b)$ (which is boundary of $\mathcal{C}$

Question: what would a multivariable (or vector calculus) FTOC look like?

For double integrals in particular, what would a FTOC look like?

$$\iint \text{"derivative"} = ?$$

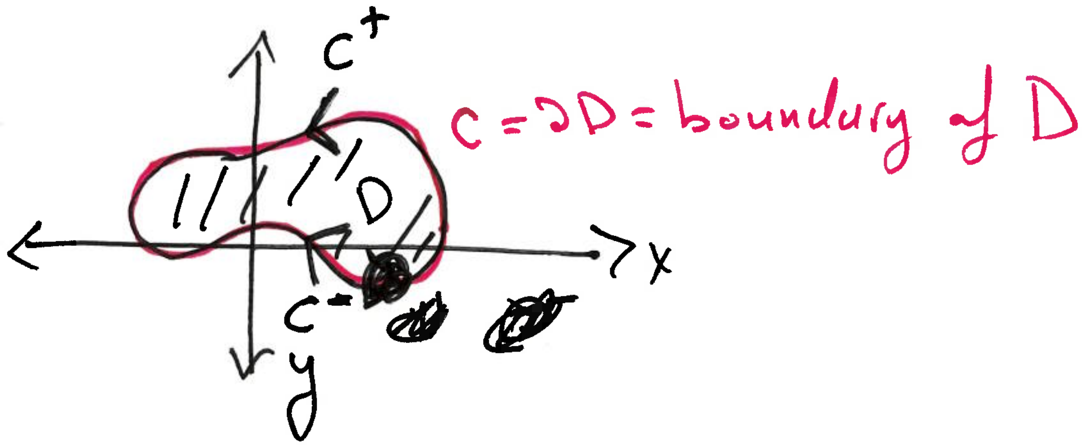
Elementary Regions and Boundaries


not closed curve

D = elementary region

$C = \partial D$ = boundary of D

C is a simple closed curve



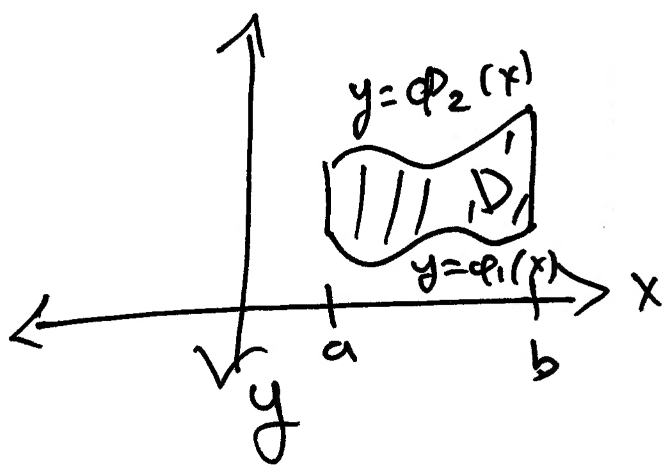
$C^+ = C$ with counterclockwise orientation

$C^- = C$ with clockwise orientation

Lemma 1: D - y-simple region and C its boundary. Let $P: D \to \mathbb{R}$ scalar function of class C^1. Then

$$-\iint_D \frac{\partial P}{\partial y}\, dx\, dy = \int_{C^+} P\, dx + 0 \cdot dy \qquad F = (P, 0)$$

Proof: First calculate left-hand side



D: $a \le x \le b$
$\varphi_1(x) \le y \le \varphi_2(x)$

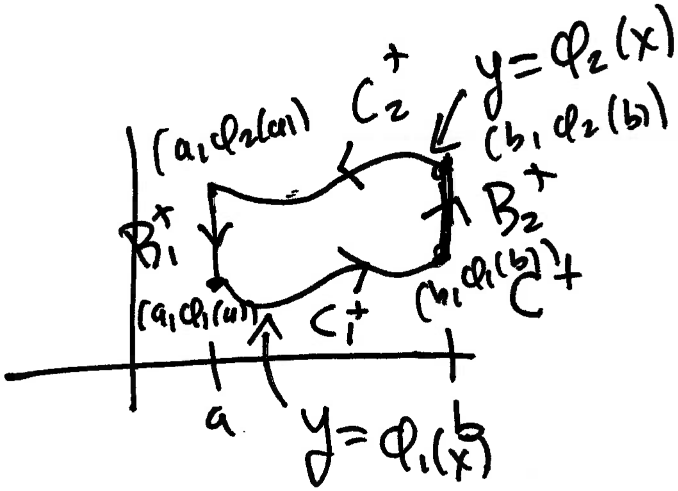
D is y-simple so we can express D as

$$\iint_D \frac{\partial P}{\partial y}\, dx\, dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\, dy \right) dx \overset{\text{FTOC}}{=} \int_a^b P(x, \varphi_2(x)) - P(x, \varphi_1(x))\, dx$$

from single variable calculus

Calculate right-hand side

$\boxed{\text{line integral where } F(x,y) = (P, 0)}$



$$\int_{C^+} P\,dx + 0\cdot dy$$

$$= \int_{C_1^+} P\,dx + \int_{B_2^+} P\,dx + \int_{C_2^+} P\,dx + \int_{B_1^+} P\,dx$$

C_1^+ is parametrized by $c(x) = (x, \varphi_1(x))$, $a \le x \le b$

$c'(x) = (1, \varphi_1'(x))$

$$\int_{C_1^+} P\,dx = \int_a^b P(x, \varphi_1(x))\,dx$$

$c(t) = (t, \varphi_1(t))$, $a \le t \le b$

$c'(t) = (1, \varphi_1'(t))$

$$= \int_a^b P(t, \varphi_1(t))\,dt.$$

C_2^- is parametrized by $c(x) = (x, \varphi_2(x))$, $a \le x \le b$

$c'(x) = (1, \varphi_2'(x))$

$$\int_{C_2^+} P\,dx = -\int_{C_1^-} P\,dx = -\int_a^b P(x, \varphi_2(x))\,dx$$

B_2^+ is parametrized by $c(y) = (b, y)$, $\varphi_1(b) \leq y \leq \varphi_2(b)$

$c'(y) = (0, 1)$

$c(t) = (b, t)$, $\varphi_1(b) \leq t \leq \varphi_2(b)$

$c'(t) = (0, 1)$

$$\int_{B_2^+} P\,dx = 0$$

$$\parallel$$

$$\int_{\varphi_1(b)}^{\varphi_2(b)} P(b,t) \cdot 0 \cdot dt = 0$$

B_1^- is parametrized by $c(y) = (a, y)$, $\varphi_1(a) \leq y \leq \varphi_2(a)$

$c'(y) = (0, 1)$

$$\int_{B_2^+} P\,dx = -\int_{B_1^-} P\,dx = 0$$

$$\int_{C^+} P\,dx = \int_{C_1^+} P\,dx + \int_{C_2^+} P\,dx + \int_{B_1^+} P\,dx + \int_{B_2^+} P\,dx = \int_a^b P(t, \varphi_1(t))\,dt - \int_a^b P(t, \varphi_2(t))\,dt$$

$$\parallel$$

$$-\iint_D \frac{\partial P}{\partial y}\,dx\,dy \quad \square$$

__Lemma 2__: D - x-simple region with boundary C.
Let $Q : D \to \mathbb{R}$ be of class C^1. Then

$$\iint_D \frac{\partial Q}{\partial x} = \int_{C^+} Q\,dy + 0 \cdot dx$$

proof is same as Lemma 1 with x and y switched.

__Green's Thm__: D - simple region, C = boundary of D
$P, Q : D \to \mathbb{R}$ of class C^1. Then

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \int_{C^+} P\,dx + Q\,dy = \int_{\partial D^+} F \cdot d\vec{s}$$

Meaning: $\iint_D$ "derivative" of $F\,dA$ $\int_{\partial D^+} F \cdot d\vec{s}$ $\quad F(x,y) = (P(x,y), Q(x,y))$